When a batch of a certain chemical is prepared, the amount of a particular impurity in the batch is a random variable with mean 4.0 grams and a standard deviation of 1.5 grams.

a) When a batch is prepared, what is the probability that the amount of impurity in the batch exceeds 3.5 grams?

b) When 50 batches are independently prepared, what is the chance that the sample average amount of impurity exceeds 3.5 grams?

Solutions:

Let $X$ denote the amount of impurity in one batch of the chemical. Let $X_1, X_2, \ldots, X_{50}$ denote a random sample of batches and let $\bar{X}$ denote the average impurity among this random sample.

a) Without knowledge of the probability distribution for $X$, there is no way to answer the first part of this question accurately. However, for instance, if we knew that $X$ has a normal distribution, we could state that

$$ P \left( \frac{X - 4.0}{1.5} > \frac{3.5 - 4.0}{1.5} \right) = P \left( Z > -0.33 \right) \approx 0.6293. $$

But, we don’t know what the distribution of $X$ is, so we can’t answer part a).

b) We know from the Central Limit Theorem that $\bar{X}$ has an approximate normal distribution with mean $\mu_{\bar{X}} = \mu_X = 4.0$ and a standard deviation $\sigma_{\bar{X}} = \sigma_X / \sqrt{n} = 1.5/\sqrt{50} \approx 0.2121$. This is true regardless of the distribution of $X$. The distribution of $X$ could be uniform, exponential, normal, or whatever for all we care. It doesn’t matter. The CLT guarantees that $\bar{X}$ has an approximate normal distribution when the sample size $n$ is large. We want $P \left( \bar{X} > 3.5 \right)$.

$$ P \left( \frac{\bar{X} - 4.0}{1.5/\sqrt{50}} > \frac{3.5 - 4.0}{1.5/\sqrt{50}} \right) \approx P \left( Z > -2.36 \right) \approx 0.9909. $$