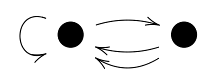
Counterexamples to the Shift Equivalence Conjecture

Nicholas Long
University of Maryland
Department of Mathematics

Preliminary Oral Examination

November 15, 2005

Matrices over $\mathbb{Z}^+$represent shifts of finite type (SFTs).

Example: If $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, then $\mathcal{G}_A =$ 

This directed graph gives rise to the SFT (X_A, σ_A).

Since SFT's are the fundamental objects of symbolic dynamics, it is natural to ask when are (X_A, σ_A) and (X_A, σ_B) conjugate.

SSE Theory

Given matrices A,B over unital semiring $\mathcal{S}$:

ESSE: Elementary Strong Shift Equivalence:

A is ESSE to B (over $\mathcal{S}$) if $\exists$ matrices R and S over $\mathcal{S}$ with $A = RS$, $B = SR$.

SSE:Strong Shift Equivalence

$A \approx_{\mathcal{S}} B$ (A is SSE to B over $\mathcal{S}$) if there is a chain of ESSE (over $\mathcal{S}$) between A and B.

Theorem [Williams] $(X_A, \sigma_A) \cong (X_B, \sigma_B)$ iff A $\approx_{\mathbb{Z}^+}$B.

But SSE over $\mathbb{Z}^+$is not known to be decidable.

SE: Shift Equivalence

$A \sim_{\mathcal{S}} B$ (A is SE to B over $\mathcal{S}$) with lag l if there exist matrices R and S over $\mathcal{S}$ with

$$RA = BR \qquad AS = SB \qquad A^l = RS \qquad B^l = SR.$$

Why use SE?

- SE over $\mathbb{Z}$ is well understood.
- Matrices over $\mathbb{Z}$ are SE (over $\mathbb{Z}$) to a non-singular matrix.
- SE over $\mathbb{Z}$ can be analyzed with number theory and linear algebra.
- SE over $\mathbb{Z}^+$is decidable.
- SE over $\mathbb{Z}$ $\Leftrightarrow$ SSE over $\mathbb{Z}$.
- In the important case of primitive matrices (mixing SFT's), SE over $\mathbb{Z}_+$ $\Leftrightarrow$ SSE over $\mathbb{Z}$.

Definition: If A is an $n \times n$ matrix over $\mathbb{Z}^+$, then

$$G_A = \lim_{A\rightarrow} \mathbb{Z}^n$$

with $G_A^+ =$ the induced positive set of G_A and $\hat{A} =$ the automorphism of G_A induced by A. $(G_A, G_A^+, \hat{A})$ is called the dimension module.

Facts:

- A $\sim_{\mathbb{Z}}$ B iff $(G_A, \hat{A}) \cong (G_B, \hat{B})$
- A $\sim_{\mathbb{Z}^+}$B iff $(G_A, G_A^+, \hat{A}) \cong (G_B, G_B^+, \hat{B})$

Example: Suppose A is a matrix over $\mathbb{Z}$ and det(A) $= \pm 1$. Then $G_A = \mathbb{Z}^n$, $\hat{A} = A$, and $A \sim_{\mathbb{Z}} B$ iff A and B are conjugate in $Gl_n(\mathbb{Z})$.

Clearly A $\approx_{\mathbb{Z}^+}$B $\Rightarrow$ A $\sim_{\mathbb{Z}^+}$B.

SE Problem : When does A $\sim_{\mathbb{Z}^+}$B $\Rightarrow$ A $\approx_{\mathbb{Z}^+}$B?

Theorem [Williams, Annals of Math. 1973]: SE over $\mathbb{Z}^+$implies SSE over $\mathbb{Z}^+$

Conjecture [Williams, Annals of Math. 1974]: SE over $\mathbb{Z}^+$implies SSE over $\mathbb{Z}^+$

Kim and Roush refuted this conjecture in 1992 for the reducible case and in 1997 for the irreducible (in fact primitive) case.

Automorphisms of σ_A

Definition: Let $Aut(\sigma_A)$ be the group of homeomorphisms that commute with the shift.

$Aut(\sigma_A)$ is very large, so consider $Aut(\widehat{A})$ the automorphisms of the dimension module, a much more tractable group to study.

Any $\phi \in Aut(\sigma_A)$ can be realized by a SSE in $\mathbb{Z}^+$, $(R_1, S_1)(R_2, S_2)...(R_k, S_k)$.

Let $\widehat{\phi}$ be the induced automorphism on $(G_A, \widehat{A})$, $\prod(\widehat{R}_i)^{\epsilon_i}$, giving a map $\rho : Aut(\sigma_A) \to Aut(\widehat{A})$. $\widehat{\phi}$ does not depend on the choice of SSE representing ϕ.

This map, ρ, is the dimension representation [Krieger]. Elements in its kernel are called inert automorphisms.

Definition: The group of (inert) automorphisms that are conjugate to an automorphism induced by a graph automorphism that fixes vertices of the directed graph are called the simple automorphisms ($Simp(\sigma_A)$).

SGCC Representation

SGCC is another very fruitful representation that grew out of studying $Aut(\sigma_A)$ action on periodic points.

If $\phi : (X_A, \sigma_A) \to (X_B, \sigma_B)$ is a conjugacy with basis x_i for σ_A-orbits of cardinality m, and similarly y_i for B, then $\phi(x_i) = (\sigma_B)^{n_i} y_{\pi(i)}$

$OS(\phi) = sign(\pi)$ (which is in $\mathbb{Z}/2\mathbb{Z}$, 0 or 1)

$GY(\phi) = \sum n_i$ which is a value in $\mathbb{Z}/m\mathbb{Z}$ and combines to

$$SGCC_m(\phi) = GY_m(\phi) + (m/2) \sum_{i<m:m/i=2^k} OS_i(\phi)$$

If $(X_A, \sigma_A) = (X_B, \sigma_B)$, then $SGCC : Aut(\sigma_A) \to \prod_m \mathbb{Z}/m\mathbb{Z}$.

SGCC will vanish for

- compositions of involutions on full shifts [Boyle and Krieger]

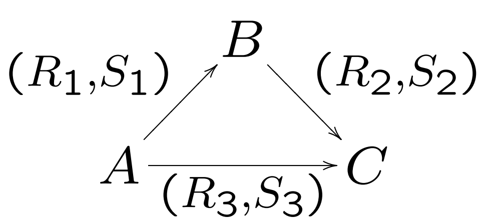
- compositions of finite order automorphisms [Fiebig]

- simple automorphisms [Nasu]

- inert automorphisms [KimRoushWagoner] (thus SGCC(ϕ) is determined by $\rho(\phi)$).

SSE Spaces [Wagoner]

Let $\mathcal{S}$ be a unital subring, $SSE(\mathcal{S})$ is the CW complex obtained from letting the 0-cells be square matrices over $\mathcal{S}$, the (oriented) 1-cells are ESSE over $\mathcal{S}$, and 2-cells attached when the triangle identities are satisfied. (A, B, and C not necessarily distinct)



$$R_1R_2 = R_3$$

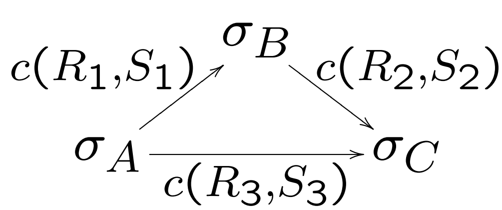
$$R_2S_3 = S_1$$

$$S_3R_1 = S_2$$

Higher cells are attached in a natural way.

For each A, we associate a specific SFT and to each edge we associate a specific conjugacy $c(R,S) : \sigma_A \to \sigma_B$.

The Triangle Identities are constructed to guarantee that the diagram of associated conjugacies will commute (up to simple automorphisms).



Wagoner shows:

$\pi_0(SSE(\mathbb{Z}^+), A) =$ SSE class of A over $\mathbb{Z}^+$

$\pi_1(SSE(\mathbb{Z}^+), A) = Aut(\sigma_A)/Simp(\sigma_A)$

$\pi_i(SSE(\mathbb{Z}^+), A) = 0$ for all $i \geq 2$

$\pi_0(SSE(\mathbb{Z}), A) =$ SSE class of A over $\mathbb{Z}$

$\pi_1(SSE(\mathbb{Z}), A) = Aut(\widehat{A})$

$\pi_i(SSE(\mathbb{Z}), A) = 0$ for all $i \geq 2$

For each 0-cell A, a basis for periodic points of size m is chosen in an uniform lexicographic way.

Given an edge $(R,S) : A \to B$ and $m \in \mathbb{N}$, $SGCC_m$ is defined for the associated conjugacy c(R,S), with respect to the given bases for the periodic points of σ_A and σ_B.

$SGCC_m$ is defined on a path of edges by summing and will agree with the original definition for a path from A to A.

The $SGCC_2$ representation agrees on SSE($\mathbb{Z}^+$) with sgc_2, a function that is polynomial in the entries of the (R,S) matrices for an ESSE.

$sgc_2(R,S) = \sum_{i<j,k>i} R_{ik}S_{ki}R_{jl}S_{lj} + \sum_{i<j,k\geq i} R_{ik}S_{kj}R_{jl}S_{li} + \sum_{i,k} \frac{R_{ik}(R_{ik}-1)}{2} S_{ki}^2$

It can be shown

- SGCC vanishes around triangles in SSE($\mathbb{Z}^+$)
- sgc vanishes around triangles in SSE($\mathbb{Z}^+$)
- sgc vanishes around triangles in SSE($\mathbb{Z}$).

Thus for a path, ρ, sgc(ρ) depends only on the homotopy class of ρ in SSE($\mathbb{Z}$).

The Δ-strategy [KimRoush and Wagoner]

Let $\mathcal{C}$ be some union of components in SSE($\mathbb{Z}^+$). $\pi_1(SSE(\mathbb{Z}), \mathcal{C})$ is the homotopy classes of paths that have initial and terminal vertices in $\mathcal{C}$.

Let Δ be a homomorphism $\pi_1(SSE(\mathbb{Z}), \mathcal{C}) \to G$, where G is an abelian group and Δ vanishes on paths in $\mathcal{C}$.

If A and B are matrices in $\mathcal{C}$ and α is a path between them in SSE($\mathbb{Z}$), then any other path between them, β, would have $\Delta(\alpha) = \Delta(\beta) + \Delta(\alpha * \beta^{-1})$

So $\Delta(\alpha)$ is a quantity, $\Delta(A, B)$, that depends only on A and B mod $\Delta(\pi_1(SSE(\mathbb{Z}), A))$. $\Delta(A, B)$ will vanish if there is a path between A and B contained in $\mathcal{C}$.

The counterexample arises from finding a Δ and matrices A and B in $\mathcal{C}$ with $\Delta(A, B) \neq 0$ for some β and $\Delta(\alpha) = 0$ for α in $\pi_1(SSE(\mathbb{Z}), A)$.

Kim and Roush found primitive matrices A and B with:

- $tr(A) = tr(A^2) = 0$
- sgc_2 vanishes on $Aut(\widehat{A})$
- a path in SSE($\mathbb{Z}$) from A to B with nonzero sgc_2

Polynomial Matrices

There is an alternate algebraic structure for studying conjugacies of SFTs that uses matrices over polynomial rings.

- Square matrix over 0,1 $\rightarrow$ vertex SFT
- Square matrix over $\mathbb{Z}^+ \rightarrow$ edge SFT
- Square matrix over t$\mathbb{Z}^+[t] \rightarrow$ route SFT

Example: Let $A = \begin{bmatrix} t^3 & 2t \\ t & 0 \end{bmatrix}$, corresponds to an edge SFT of

$\mathcal{G}_A =$

Instead of using ESSE equivalences to classify conjugacies, we can use multiplications of I-A by elementary matrices that satisfy some positivity conditions.

$(I-A)E = I-B$ or $E(I-A) = I-B$ with E being basic elementary and A, B over $t\mathbb{Z}^+[t]$.

In general one needs to work with matrices over $\mathbb{Z}^+[t]$which satisfy the No Z-Cycles Condition, NZC, saying that A(t = 0) has no periodic point.

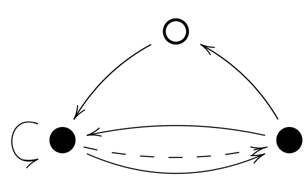
K-Theory can then be used to help analyze the different ways that elementary matrices can be used to form these conjugacies.

Positive K-Theory

Let A be a matrix over $\mathbb{Z}^+[t]$, A defines a SFT through the path construction using essential vertices.

Example: Let $A = \begin{bmatrix} t & t+t^2 \\ t+1 & 0 \end{bmatrix}$, then

$\mathcal{G}_A =$



All of this construction is done in the presence of the NZC (A(0) has no periodic points).

Let A, B be matrices over $\mathbb{Z}^+[t]$.

Classification Theorem [Boyle and Wagoner]: (X_A, σ_A) and (X_B, σ_B) are topologically conjugate iff there are a sequence of positive row and column operations over $\mathbb{Z}^+[t]$connecting I - A and I - B.

Conjugacy Theorem[Boyle and Wagoner]: Every topological conjugacy from (X_A, σ_A) to (X_B, σ_B) arises from some sequence of positive row and column operations over $\mathbb{Z}^+[t]$connecting I - A and I - B.

Notice that if A is a matrix over $\mathbb{Z}^+$, then tA (a matrix over $\mathbb{Z}^+[t]$) defines a conjugate SFT.

Polynomial Strong Shift Equations (PSSE) give the positive operations from $I-tA = I-tRS$ to $I-B = I-tSR$ using a (R,S) edge in SSE($\mathbb{Z}^+$)
.

$$\begin{pmatrix} I-tRS & 0 \\ -tS & I \end{pmatrix}\begin{pmatrix} I & 0 \\ tS & I \end{pmatrix} = \begin{pmatrix} I-tRS & 0 \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -R \\ -tS & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -tS & I-tSR \end{pmatrix}$$

$$\begin{pmatrix} I & -R \\ -tS & I \end{pmatrix}\begin{pmatrix} I & R \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -tS & I-tSR \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ tS & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -tS & I-tSR \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I-tSR \end{pmatrix}$$

These equations pointed toward using K-theory to distinguish ‘different’paths of positive operations.

The first several K groups of a unital ring $\mathcal{S}$:

K_0: Formal differences of isomorphism classes of $\mathcal{S}$-modules, measures dimension of $\mathcal{S}$-modules.

K_1: E($\mathcal{S}$) is the subgroup of GL($\mathcal{S}$) generated by elementary matrices. $K_1(\mathcal{S}) = GL(\mathcal{S})^{ab} = GL(\mathcal{S})/E(\mathcal{S})$ gives the abelian invariants of matrices.

K_2: $K_2(\mathcal{S}) = ker(St(\mathcal{S}) \to E(\mathcal{S}))$ measures the number of 'different 'ways of reducing a matrix using elementary operations.

Wagoner constructed a function Φ_{2m} that satisfies the Δ strategy (and happens to agree with sgc_2 when m = 1). Φ_{2m} is defined by taking a path in SSE($\mathbb{Z}^+$) from A to B and using the following steps:

1) If A is a matrix over $\mathbb{Z}^+$, then tA (a matrix over $\mathbb{Z}^+[t]$) defines a conjugate SFT.

2) PSSE equations are used to obtain the positive operations that go between I-tA and I-tB.

$$E_L(\gamma)(I - tA)E_R(\gamma) = I - tB$$

3) If M is a matrix over $\mathbb{Z}^+$and $Tr(M) = 0$, then $M_{ii} = 0$ and
$L_M(I - tM)R_M = (I - t^2M')$ for
$L_m = \Pi_{j=1}^{n-1}\Pi_{i=j+1}^{n} e_{ij}(tM_{ij})$
$R_m = \Pi_{i=1}^{n-1}\Pi_{j=i+1}^{n} e_{ij}(tM_{ij})$.

4) If Tr(A) = Tr(B) = 0 then we can get the equation

$$L_B E_L(\gamma) L_A^{-1}(I - t^2A')R_A^{-1}E_R(\gamma)R_B = I - t^2B'$$

And over the dual numbers, $\mathbb{Z}[t]/(t^2)$, we get the equation

$$L_B E_L(\gamma) L_A^{-1} R_A^{-1} E_R(\gamma) R_B = I$$

which gives an element of K_2.

And this gives us Φ_2 from $\pi_1(SSE(\mathbb{Z}), SSE_2(\mathbb{Z}^+))$ to $K_2(\mathbb{Z}[t]/(t^2), (t))$. It can be shown that this function vanishes on SSE($\mathbb{Z}^+$) and around SSE($\mathbb{Z}$) triangles using only Steinberg relations and the vanishing trace consequences.

Wagoner used Φ_2 as his function in the Δ-strategy.

Future Work

Although William's conjecture has been refuted there is still the problem of deciding conjugacy (SSE over $\mathbb{Z}^+$).

Question 1: Suppose A and B are primitive in $\mathbb{Z}^+$and there is a path in SSE($\mathbb{Z}$) with the sgc of the path vanishing. Does this imply that there is a deformation of the path to a path in SSE($\mathbb{Z}^+$)?

There are several intermediate steps to answering this question which include how to finitely represent sgc of a path.

In the positive K-theory setting,

Question 2: Suppose A and B are matrices over $\mathbb{Z}^+[t]$satisfying NZC and there is a path of elementary (not necessarily positive) operations between I-A and I-B. Is there an obstruction on the path that would imply that there is a path of positive equivalences?

Question 3: Can sgc be computed directly from U (I-A) V = I-B ?

There are also further questions about the shift equivalence problem for other kinds of rings. Boyle and Handelman showed that the shift equivalence problem is true for Dedekind domains. Does this generalize to rings with finite cohomological dimension? What about Laurent polynomial rings in several variables?