Polynomial Matrices in Symbolic Dynamics

Nicholas Long

21 January 2007

Abstract

In this paper, we will discuss the role that polynomial matrices play in symbolic dynamics. We will introduce shift spaces and shifts of finite type while discussing how they can be represented by various types of directed graphs and the corresponding matrices. We will then discuss several invariants of shift spaces and how they are linked to the different representations. Finally, we will discuss how polynomial matrices have been used in several useful constructions.

Introduction

A discrete dynamical system is a set with an integer action on the set. This action can be described as the iterations of a function on the set. Typically, the function is continuous on a nice topological space, like a manifold.

Symbolic dynamics is a branch of discrete dynamical systems that deals with the properties of shift spaces. A full shift is the space of all possible bi-infinite sequences of letters from a given alphabet, $\mathcal{A}$. For example, the full 2-shift ($X_{[2]}$) is the space of all possible bi-infinite sequences of 0's and 1's. A point in the full 2-shift is $...x_{-2}x_{-1}x_0.x_1x_2x_3... = x \in X_{[2]}$ where x_i is either 0 or 1 and i is indexed by the integers. There is a natural map, σ, called the shift map that shifts the sequence, $\sigma(x) = ...x_{-2}x_{-1}.x_0x_1x_2x_3....$ A map f on a shift space is called shift commuting or shift invariant if $f \circ \sigma_X = \sigma_X \circ f$.

A word is just a finite combination of symbols from the alphabet $\mathcal{A}$. In a full shift, there are no forbidden words since any combination of symbols is allowed. If $\mathcal{F}$ is a list of forbidden words from an alphabet $\mathcal{A}$, then $X_{\mathcal{F}}$ is set of points from the full shift on $\mathcal{A}$ where no word from $\mathcal{F}$ appears anywhere. $X_{\mathcal{F}}$ is a shift invariant subset of the full shift called a shift space. The same shift space can be defined by many different lists of forbidden words. The set $\mathcal{F}$ can be a finite or infinite list but is always countable since there is a countable number of possible words. If $X_{\mathcal{F}}$ can be constructed with a finite list of forbidden words, then we call $X_{\mathcal{F}}$ a shift of finite type or SFT.

Representations of SFTs

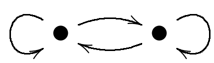
Since the forbidden word construction of a SFT is not unique, we will discuss a couple other representations of SFTs that have better algebraic properties. The basic idea for these representations is to use the correspondence between bi-infinite walks in a directed graph and the bi-infinite sequences in the SFT.

A directed graph is a finite set of vertexes with a finite set of edges such that each edge has an initial and terminal vertex which determines its direction. For each directed graph there exists an adjacency matrix, A, defined by $(A)_{ij}$ = the number of edges from vertex i to vertex j. Note that since each edge has a direction, the adjacency matrix is not necessarily symmetric. So any given matrix over the non-negative integers defines a directed graph. A bi-infinite walk on a directed graph is a bi-infinite sequence of edges such that the terminal vertex of an edge is the initial vertex of the next edge.

Edge Representations

If each edge is assigned a symbol, then the set of all possible bi-infinite sequences of edges defines a shift space which is also a SFT. This representation is called an edge shift representation since the edges traversed are recorded to make the symbolic sequence. Edge shift representations, or edge shifts, are useful because all SFTs can be represented by an edge representation.This means that a SFT can be represented by a square matrix with entries in the semi-ring of the non-negative integers, $\mathbb{Z}^+$. Neither directed graphs or matrices over $\mathbb{Z}^+$ give rise to unique representations for a SFT. Consider the matrices $A = [2]$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which have associated graphs of

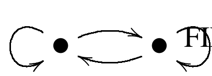
$\mathcal{G}_A$ = 

$\mathcal{G}_B$ = 

Both graphs represent the full 2-shift but are based on matrices of different sizes. The biggest advantage of using an edge representation rather than the forbidden word construction is that linear algebra techniques can be used on the adjacency matrix to analyze the SFT.

Path Representations

Edge representations of SFTs correspond to square matrices over $\mathbb{Z}^+$. Square matrices over $t\mathbb{Z}^+[t]$ can also represent a SFT. Let A be an *nxn* matrix over $t\mathbb{Z}^+[t]$. From A we wish to construct a directed graph that will define the corresponding SFT. The graph produced by A will have at most n essential vertexes and several other non-essential vertexes. A vertex is non-essential if there is one incoming edge and one out going edge. The graph defined by A starts with the n essential vertexes but we will add edges and non-essential vertexes based on the entries of A. If $A_{1,2} = t + 2t^3$, then from essential vertex 1 to essential vertex 2 we add one edge of length 1 (no additional non-essential vertexes) and 2 paths of length 3. A path of length n is a sequence of n edges that begin at an essential vertex, go through n-1 non-essential vertexes, and terminates at an essential vertex. So if $A = \begin{bmatrix} t & 2t^2 \\ t^4 & 3t \end{bmatrix}$, then the directed graph that the path representation produces is

$\mathcal{G}_A$ =  FINISH THIS GRAPH LATER

The adjacency matrix for $\mathcal{G}_A$ would be 7x7 since the edge representation does not distinguish between essential and non-essential vertexes. Using matrices over $t\mathbb{Z}^+[t]$ and the corresponding path construction allows for a more compact representations of graphs. There are also additional algebraic tools for matrices over a polynomial ring that can be used for finer analysis of SFTs which we will discuss in the last section of the paper.

If B is an adjacency matrix for an edge representation, then $C = t * B$ defines the same directed graph using the path construction. This will enable us to start with an edge representation and convert to a path representation so that we can then use the tools available for polynomial matrices. In general, the conversion from a path representation to an edge representation involves building the directed graph form the path construction and then creating the adjacency matrix from this graph. Both representations are equivalant and which ever one offer teh most convienience can be used. Let X_A be the SFT defined by A, a matrix over either $t\mathbb{Z}^+[t]$ or $\mathbb{Z}^+$

Invariants of SFTs

A SFT is mixing if there exists a natural number N, such that for each pair of allowed words, u and v, there is a word w of length n for each $n \geq N$ with uwv is an allowed word. An matrix representation of a MSFT must be primitive. A matrix over the integers, B, is primitive if there is some $n \in \mathbb{N}$ such that $(B^n)_{i,j} > 0$ for all i,j entries. Two shift spaces X and Y are called conjugate if there is a shift-commuting isomorphism between them.

A fundamental open problem in symbolic dynamics is the classification of mixing SFTs up to conjugacy. We have seen that the matrix and graph representations for MSFTs are not unique, so we must look at other invariants.

One particularly nice feature of shift spaces is that periodic points of the shift are easily found and for SFT they are easily counted as well. Periodic points a shift space can be seen as the infinite repetition of a closed path in the directed graph or as the infinite repetition of an word that is allowed to both precede and follow itself in the shift space. (...000.000...), (...0101.0101...), and (...0111.01110111...) are examples of points of the full 2-shift that are least period 1, 2, and 4 respectively. Let $Per(X,n)$ denote the set of points of X of period n. The zeta function keeps track of the number of periodic points as a power series.

$\zeta_A(t) = exp(\Sigma_{n=0}^{\infty} \frac{|Per(X,n)|}{n} t^n)$

The zeta function is actually related to the characteristic polynomial of any representing matrix. $\zeta_A(t) = \frac{1}{t^r \chi_A(t^{-1})} = \frac{1}{det(ID - tA)}$, where $\chi_A(t)$ is the characteristic polynomial of the r x r matrix A. This means that all of the different matrix representations for a SFT must have the same non-zero eigenvalues, which is an easily computed invariant.

Another invariant of conjugacy is entropy. The entropy of a shift space is defined by $h(X) = \frac{lim}{n \to \infty} \frac{1}{n} log|B_n(X)|$, where $B_n(X)$ is the set of allowed words in X of length n. The entropy of a shift space measures the exponential rate at which the number of allowed words increases. The Perron-Frobenius Theorem tells us that for a MSFT, the entropy is the eigenvalue with largest modulus. Entropy is also easily computed but not a fine enough invariant to completely describe conjugacy.

It is easy to see that both the periodic point counts and the entropy can be easily calculated for matrices over $\mathbb{Z}$ or $\mathbb{Z}^+$, but when using the path construction we have matrix elements over $t\mathbb{Z}^+[t]$. Computing the zeta function or entropy comes from computing the characteristic polynomial of an integral matrix so we need just to relate the path construction to the characteristic polynomial of the adjacency matrix associated to it. If A is a matrix over $\mathbb{Z}^+$ and B is a matrix over $t\mathbb{Z}^+[t]$ such that they are presentations for the same directed graph, then $det(I - tA) = det(I - B)$. So simply taking a determinant of a polynomial matrix allows us to calculate the information needed for the zeta function and entropy.

The periodic points and the entropy are not the only invariants of a SFT but are simple and easily computed.

Constructions Using Polynomial Matrices

Different constructions using polynomial matrices and the path construction have been useful in several important results over the past 15 years. We will discuss how elementary matrix operations on polynomial matrices can be used to find all possible conjugacies of a fixed SFT. Elementary operations are also used in order to recode a polynomial matrix into convienent forms.

Let A be a matrix over $t\mathbb{Z}^+[t]$ and let $x = A_{i,j}$ for $i \neq j$. Then let $E_{ij}(b)$ be the matrix that is the identity except for the i, j entry which is b. If $Id - B = [E_{ij}(b)(Id - A)]$ or $Id - B = [(Id - A)E_{ij}(b)]$, then B defines a polynomial matrix such that X_A is conjugate to X_B. For example, if $A = \begin{bmatrix} t & 2t^2 \\ t^4 & 3t \end{bmatrix}$, $b = t^4$, then $B = \begin{bmatrix} 1-t-2t^6 & 2t^2 \\ 3t^5 & 3t \end{bmatrix}$.

This means that we can construct many different conjugate SFTs just by elementary operations. Boyle and Wagoner [BW] show that all such conjugacies arise from some sequence of these elementary operations.

If A and B are matrices over $t\mathbb{Z}^+[t]$, then

Classification Theorem: X_A and X_B are topologically conjugate iff there are a sequence of positive row and column operations over $t\mathbb{Z}^+[t]$ connecting I - A and I - B.

Conjugacy Theorem: Every topological conjugacy from (X_A, σ_A) to (X_B, σ_B) arises from some sequence of positive row and column operations over $t\mathbb{Z}^+[t]$ connecting I - A and I - B.

So we know all conjugacies arise as some combination of elementary operations, but the obstruction to classifying SFTs is how to realize what the sequence of elementary operations is. While this problem of realization has not be solved, there are several proposals that use invariants of polynomial matrices to invent a decision procedure for when two MSFT are conjugate.

Let us return to the example given above where, $A = \begin{bmatrix} t & 2t^2 \\ t^4 & 3t \end{bmatrix}$, $b = t^4$, and

$B = \begin{bmatrix} 1-t-2t^6 & 2t^2 \\ 3t^5 & 3t \end{bmatrix}$. It is an interesting and useful fact that when we multiplied by an elementary matrix, the resulting matrix has a higher order term in the 1,2 position. The multiplication of the elementary matrix allowed us to clear a lower order term in favor of a higher order term in its place, and a low order term on the diagonal. When this clearing process is repeated for all off-diagonal terms below some degree m, the resulting matrix will have all off diagonal terms with degree larger than m and the diagonal will contain several new terms of degree less than or equal to m. Remember that terms on the diagonal correspond to periodic points of the shift. The clearing process will allow us to recode a matrix to a form that has low degree terms only as periodic points. This is a useful technique to exploit if we wish to extend some property from the periodic points to the entire shift space. For example, Kim and Roush used this clearing process to prove thier p-fold covering theorem.

Theorem: If X is a MSFT, then there exists a MSFT Y, such Y that is a p-fold regular cover of X and Y is shift equivalent to X iff the periodic points of X, Per(X), have a free $\mathbb{Z}/p$ action and Per(X) is isomorphic to its quotient by the free $\mathbb{Z}/p$.

The power of this theorem comes from the fact that the existance of a covering space can be confirmed by examining the subsystem of periodic points.