

VERTEX COALGEBRAS, COMODULES, COCOMMUTATIVITY AND COASSOCIATIVITY

KEITH HUBBARD

ABSTRACT. We introduce the notion of vertex coalgebra, a generalization of vertex operator coalgebras. Next we investigate forms of cocommutativity, coassociativity, skew-symmetry, and an endomorphism, D^* , which hold on vertex coalgebras. The former two properties require grading. We then discuss comodule structure. We conclude by discussing instances where graded vertex coalgebras appear, particularly as related to Primc's vertex Lie algebra and (universal) enveloping vertex algebras.

1. INTRODUCTION

Since its inception in the 1980s when Borchers first introduced the precise notion of a vertex algebra [B], the theory of vertex algebras has played an ever expanding role throughout mathematics. Initially motivated by the string theoretic paper [BPZ], the Monster group (the largest sporadic finite simple group) (cf. [FLM]) and infinite-dimensional representation theory [FK, FLM, S], vertex algebras now have known connections to modular functions and modular forms (cf. [Z, H2]), Riemann surfaces (cf. [H]), Calabi-Yau manifolds (cf. [Fri, GSW, GMS]), infinite-dimensional integrable systems (cf. [LW, FF1, FF2, FF3, DJM]), knot and three-manifold invariants (cf. [J, RT, W]), and elliptic cohomology (cf. [GMS, GM, ST, Bo]). Modules of vertex algebras play a central role in many of these connections.

The notion of vertex operator algebras (VOA) is a refinement of a the notion of vertex algebra which also encode a conformal structure [FLM]. (The conformal structure was actually satisfied by the original example used to motivate the notion of vertex algebras [B].) The dualization of the VOA structure to that of a vertex operator coalgebra (VOC) structure has recently been described through the geometry underlying both structures ([Hub1], [Hub2]). In this paper we generalize the notion of vertex operator coalgebra to the notion of vertex coalgebra, investigate vertex coalgebra properties, explore how graded vertex coalgebra structures may be generated from (and on) graded vertex algebras, and also explore vertex comodule structures.

Both vertex algebras and VOAs satisfy a natural skew-symmetry as well as several formal derivative properties. Additionally, the multiplication operators in these objects satisfy weakened versions of commutativity (also called locality) and associativity. These properties not only prove valuable in understanding the structures of vertex algebras and VOAs but may also be used to reformulate the definitions of vertex algebra and VOA in such a way that it is easier to establish when such a structure exists. Motivated by the vertex (operator) algebra case, our examination reveals analogous properties: formal derivatives and skew-symmetry for vertex

coalgebras, and cocommutativity and coassociativity properties for graded vertex coalgebras. All of these properties extend naturally to VOCs and in most cases to comodules, allowing us to generate a number of reformulations of the axioms of vertex coalgebras, graded vertex coalgebras, VOCs and their comodules.

Classically, one of the most common incarnations of a coalgebra structure is on the enveloping algebra of a Lie algebra. We describe a way to induce this coalgebra structure from the algebra structure of the enveloping algebra. We then parallel this construction for vertex Lie algebras and Primc's enveloping vertex algebra [P]. Our construction, which requires imposing a grading on the vertex Lie algebra, endows the enveloping vertex algebra with a graded vertex coalgebra structure. This is a valuable step in the search for a *general vertex bialgebra*, which subsumes the notion of vertex algebra and vertex coalgebra in a compatible fashion.

The author thanks his advisor, Katrina Barron, for her insight and guidance. He also gratefully acknowledges the financial support of the Arthur J. Schmidt Foundation.

2. DEFINITIONS AND ALGEBRAIC PRELIMINARIES

We begin, as most papers in vertex algebras, by regurgitating the required definitions and equations from the calculus of formal variables. The expositions of [FLM], [K] and [LL] subsume the discussion below. All vector spaces will be over a field \mathbb{F} of characteristic 0. We will consider x , x_0 , x_1 and x_2 commuting formal variables throughout, and define the "formal (or Dirac) δ -function" to be

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

Given n an integer, $(x_1 \pm x_2)^n$ will be understood to be expanded in nonnegative integral powers of x_2 . Note that $\delta\left(\frac{x_1 - x_2}{x_0}\right)$ is a formal power series in nonnegative powers of x_2 (cf. [FHL], [LL]). The formal residue, ' Res_x ,' refers to the coefficient of the negative first power of x in the term that follows. We will use the fact that

$$(2.1) \quad Res_{x_1} x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) = 1.$$

Several basic properties of the δ -function bear mentioning. First, given a formal Laurent series $X(x_1, x_2) \in (\text{Hom}(V, W))[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$ with coefficients which are homomorphisms from a vector space V to a vector space W , if $\lim_{x_1 \rightarrow x_2} X(x_1, x_2)$ exists (i.e. when $X(x_1, x_2)$ is applied to any element of V , setting $x_1 = x_2$ leads to only finite sums in W) we have

$$(2.2) \quad \delta\left(\frac{x_1}{x_2}\right) X(x_1, x_2) = \delta\left(\frac{x_1}{x_2}\right) X(x_2, x_2).$$

Second, we know that

$$(2.3) \quad x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right),$$

and third

$$(2.4) \quad x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right).$$

We will also make use of the following equation from Proposition 2.3.6 in [LL]:

$$(2.5) \quad \frac{\partial}{\partial x_2} x_2^{-1} \delta \left(\frac{x_1}{x_2} \right) = (x_1 - x_2)^{-2} - (-x_2 + x_1)^{-2}.$$

The final ‘usual suspect’ in the onslaught of equations we will not prove is Taylor’s Theorem, which states that given V a vector space and $f(x) \in V[[x, x^{-1}]]$,

$$(2.6) \quad e^{x_0 \frac{\partial}{\partial x}} f(x) = f(x + x_0),$$

where $e^{x_0 \frac{\partial}{\partial x}}$ is a formal exponential.

For notational ease, given any vector space V , let the linear map $T : V \otimes V \rightarrow V \otimes V$ be the transposition operator defined by $T(u \otimes v) = v \otimes u$ for all $u, v \in V$.

2.1. The notion of vertex coalgebra. The following description of a vertex coalgebra is the generalization of a vertex operator coalgebra obtained by omitting the conformal structure (i.e. the representation of the Virasoro algebra) on the underlying vector space.

Definition 2.1. *A vertex coalgebra consists of a vector space V , together with linear maps*

$$\mathcal{A}(x) : V \mapsto (V \otimes V)[[x, x^{-1}]],$$

$$c : V \mapsto \mathbb{F},$$

called the coproduct and the covacuum map, respectively, satisfying the following axioms for all $v \in V$:

1. *Left Counit:*

$$(2.7) \quad (c \otimes Id_V) \mathcal{A}(x)v = v$$

2. *Cocreation:*

$$(2.8) \quad (Id_V \otimes c) \mathcal{A}(x)v \in V[[x]] \quad \text{and}$$

$$(2.9) \quad \lim_{x \rightarrow 0} (Id_V \otimes c) \mathcal{A}(x)v = v$$

3. *Truncation:*

$$(2.10) \quad \mathcal{A}(x)v \in (V \otimes V)((x^{-1}))$$

4. *Jacobi Identity:*

$$(2.11) \quad x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)(T \otimes Id_V) \\ (Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2).$$

The operator $\mathcal{A}(x)$ is linear so that, for example, $(Id_V \otimes \mathcal{A}(x_2))$ acting on the coefficients of $\mathcal{A}(x_1)v \in (V \otimes V)[[x_1, x_1^{-1}]]$ is well defined. Notice also, that when each expression is applied to any element of V , the coefficient of each monomial in the formal variables is a finite sum by Equation (2.10). We denote this vertex coalgebra by V , or by (V, \mathcal{A}, c) when necessary.

2.2. Formal derivatives and skew-symmetry on vertex coalgebras. A reasonable (and fruitful) avenue of investigation is to question the effect of applying a formal derivative, d/dx , to the comultiplication operator \mathcal{A} of a vertex coalgebra. The following proposition gives a convenient description.

Proposition 2.2. *Let (V, \mathcal{A}, c) be a vertex coalgebra and $D^* : V \rightarrow V$ be the linear map defined to be*

$$(2.12) \quad D^* = Res_x x^{-2}(Id_V \otimes c)\mathcal{A}(x).$$

Then

$$(2.13) \quad (D^* \otimes Id_V)\mathcal{A}(x) = \frac{d}{dx}\mathcal{A}(x).$$

Proof. First, we see from the Jacobi identity (2.11), as well as (2.3), that

$$(2.14) \quad Res_{x_0} x_0^{-2} x_1^{-1} \delta\left(\frac{x_2+x_0}{x_1}\right)(\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2) \\ = (x_1-x_2)^{-2}(Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1) - (x_2-x_1)^{-2}(T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2).$$

We then use (2.12), (2.1), (2.14), (2.7), (2.5) and (2.2) in succession to show that

$$(D^* \otimes Id_V)\mathcal{A}(x_2) = Res_{x_0} x_0^{-2}(Id_V \otimes c \otimes Id_V)(\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2) \\ = Res_{x_1} Res_{x_0} x_0^{-2}(Id_V \otimes c \otimes Id_V) x_1^{-1} \delta\left(\frac{x_2+x_0}{x_1}\right)(\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2) \\ = Res_{x_1}(Id_V \otimes c \otimes Id_V) \left((x_1-x_2)^{-2}(Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1) \right. \\ \left. - (x_2-x_1)^{-2}(T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2) \right) \\ = Res_{x_1} \left((x_1-x_2)^{-2}\mathcal{A}(x_1) - (x_2-x_1)^{-2}\mathcal{A}(x_1) \right) \\ = Res_{x_1} \frac{\partial}{\partial x_2} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right)\mathcal{A}(x_1) \\ = Res_{x_1} \frac{\partial}{\partial x_2} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right)\mathcal{A}(x_2) \\ = \frac{d}{dx_2}\mathcal{A}(x_2).$$

□

There is also a convenient way to describe the cocreation axiom in terms of the operator D^* . First, we note that combining (2.7) with either (2.12) or (2.13) immediately implies that

$$(2.15) \quad cD^* = 0.$$

Exponentiation of (2.13), then applying Taylor's Theorem (2.6) implies

$$(2.16) \quad \begin{aligned} (e^{x_0 D^*} \otimes Id_V) \mathcal{A}(x) &= e^{x_0 \frac{\partial}{\partial x}} \mathcal{A}(x) \\ &= \mathcal{A}(x + x_0). \end{aligned}$$

Composing both sides of (2.16) with the map $Id_V \otimes c$ gives us a series in only non-negative powers of x so we set $x = 0$ and conclude by (2.9) that

$$(2.17) \quad e^{x_0 D^*} = (Id_V \otimes c) \mathcal{A}(x_0).$$

Equation (2.17) also implies the cocreation axiom, meaning that the two are equivalent in the presence of the other axioms.

The operator D^* has a third important application as well. Vertex coalgebras possess a natural skew-symmetry via D^* similar to the case for vertex algebras. More precisely:

Proposition 2.3. *Given V a vertex coalgebra and D^* as in (2.12), we have*

$$(2.18) \quad T\mathcal{A}(x) = \mathcal{A}(-x)e^{x D^*}.$$

Proof. By two applications of the Jacobi identity (2.11), we have

$$(2.19) \quad \begin{aligned} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (T\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2) \\ &= x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (T \otimes Id_V) (Id_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_1) \\ &\quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) (Id_V \otimes \mathcal{A}(x_1)) \mathcal{A}(x_2) \\ &= x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) (\mathcal{A}(-x_0) \otimes Id_V) \mathcal{A}(x_1). \end{aligned}$$

But by (2.2) and (2.16) the right-hand side of (2.19) is equal to

$$\begin{aligned} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) (\mathcal{A}(-x_0) \otimes Id_V) \mathcal{A}(x_2 + x_0) \\ = x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) (\mathcal{A}(-x_0) e^{x_0 D^*} \otimes Id_V) \mathcal{A}(x_2). \end{aligned}$$

Taking Res_{x_1} and using (2.3) we have

$$(T\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2) = (\mathcal{A}(-x_0) e^{x_0 D^*} \otimes Id_V) \mathcal{A}(x_2).$$

Finally, apply $(Id_V \otimes Id_V \otimes c)$ to both sides and note that by cocreation we may set $x_2 = 0$, getting (2.18). \square

Via skew-symmetry we are now able to interpret the derivative d/dx of \mathcal{A} in a second way, which might be called a D^* -bracket formula for vertex coalgebras in view of its similarity to the D -bracket formula for vertex algebras.

Proposition 2.4. *Let V be a vertex algebra and let D^* be as in (2.12). Then*

$$(2.20) \quad \frac{d}{dx} \mathcal{A}(x) = \mathcal{A}(x)D^* - (Id_V \otimes D^*)\mathcal{A}(x).$$

Proof. Using (2.18), the product rule, (2.13), and finally reapplying (2.18),

$$\begin{aligned} \frac{d}{dx} \mathcal{A}(x) &= \frac{d}{dx} \left(T\mathcal{A}(-x)e^{xD^*} \right) \\ &= T \left(\frac{d}{dx} \mathcal{A}(-x) \right) e^{xD^*} + T\mathcal{A}(-x) \frac{d}{dx} \left(e^{xD^*} \right) \\ &= -T(D^* \otimes Id_V)\mathcal{A}(-x)e^{xD^*} + T\mathcal{A}(-x)e^{xD^*} D^* \\ &= -(Id_V \otimes D^*)\mathcal{A}(x) + \mathcal{A}(x)D^*. \end{aligned}$$

\square

Exponentiating the D^* -bracket formula (2.20) yields

$$(2.21) \quad (Id_V \otimes e^{-x_0 D^*})\mathcal{A}(x)e^{x_0 D^*} = e^{x_0 \frac{d}{dx}} \mathcal{A}(x)$$

and applying Taylor's Theorem we have

$$(2.22) \quad (Id_V \otimes e^{-x_0 D^*})\mathcal{A}(x)e^{x_0 D^*} = \mathcal{A}(x + x_0).$$

2.3. Graded vertex coalgebras. Although our description of vertex coalgebra parallels Borchers' original definition of vertex algebra [B], some recent definitions have included a graded vector space (cf. [F], [FB]). Similarly, our focus will be primarily on *graded vertex coalgebras*.

Let $V = \coprod_{k \in \mathbb{Z}} V_{(k)}$ be a \mathbb{Z} -graded vector space such that $V_{(k)} = 0$ for $k \ll 0$. Any vector $v \in V_k$, $k \in \mathbb{Z}$, is said to be *homogeneous* and have weight k . We denote the graded dual space of V by

$$V' = \prod_{k \in \mathbb{Z}} V_{(k)}^* = \prod_{k \in \mathbb{Z}} \text{Hom}(V_{(k)}, \mathbb{F}),$$

the algebraic closure of V by

$$\overline{V} = \prod_{k \in \mathbb{Z}} V_{(k)} = (V')^*,$$

and the natural pairing of V' with \overline{V} by $\langle \cdot, \cdot \rangle$. This pairing is also applied to V since it may be viewed as a natural subspace of \overline{V} . The n -th tensor product of V , denoted $V^{\otimes n}$, is still a \mathbb{Z} -graded vector space (where $v \in V_{k_1} \otimes \dots \otimes V_{k_n}$ has weight $k_1 + \dots + k_n$), and $V^{\otimes n}$ inherits finite-dimensional homogeneous subspaces if V has them. Thus $(V^{\otimes n})'$, $\overline{V^{\otimes n}}$ and $\langle \cdot, \cdot \rangle : (V^{\otimes n})' \times \overline{V^{\otimes n}} \rightarrow \mathbb{F}$ are defined as above.

Definition 2.5. Let $V = \coprod_{k \in \mathbb{Z}} V_{(k)}$ be a \mathbb{Z} -graded vector space such that $V_{(k)} = 0$ for $k \ll 0$. A vertex coalgebra structure on V will be said to be graded if, given the map $\mathcal{A}(x) : V \mapsto (V \otimes V)[[x, x^{-1}]]$ and $v \in V_{(r)}$ for $r \in \mathbb{Z}$,

$$\mathcal{A}(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v)x^{-k-1}$$

where $\Delta_k(v) \in (V \otimes V)_{(r+k+1)}$ for each $k \in \mathbb{Z}$. We will say a vertex coalgebra is finite dimensionally graded if it is a graded vertex coalgebra such that for all $k \in \mathbb{Z}$ we know $\dim V_{(k)} < \infty$.

In [H], any Virasoro module that satisfies the condition $V_{(k)} = 0$ for k sufficiently small is called ‘positive energy module’. So what is called a ‘graded vertex coalgebra’ here might equally well be called a ‘positive energy graded vertex coalgebra.’ Notice that vertex operator coalgebras are finite dimensionally graded vertex coalgebras by definition [Hub1].

Understanding the family of comultiplication operators $\Delta_k : V \rightarrow V \otimes V$ is useful. Already we have a convenient way of describing the operator D^* of Section 2.2 as

$$D^* = (Id_V \otimes c)\Delta_{-2}.$$

We may view each operator Δ_k as a weight $k+1$ operator in the sense that it maps the r -th weight space of V to the $r+k+1$ -st weight space of $V \otimes V$ for each $r \in \mathbb{Z}$. Hence D^* is a weight -1 operator.

2.4. Cocommutativity properties. When investigating non-associative or non-commutative algebras, a natural question to ask is ‘How close are they to being associative or commutative?’. The same question has been studied in detail for vertex (operator) algebras (cf. [FHL], [K], [LL]). In general the multiplication operator $Y(\cdot, x) : V \rightarrow (\text{End } V)((x))$ does not commute or associate with itself but does satisfy conditions called ‘weak commutativity’ (or locality) and ‘weak associativity’. To obtain similar results for vertex coalgebras requires the use of grading and correlations functions. Here we fix a graded vertex coalgebra (V, \mathcal{A}, c) for consideration.

It might be useful to pause briefly and point out why we do not consider commutativity and cocommutativity on elements, i.e. $a \cdot b = b \cdot a$ and $\Delta a = T\Delta a$. Commutativity or cocommutativity on elements carries no implication about the associativity or coassociativity of the elements. However, if we generalize slightly and consider (left) commutativity of the multiplication operator m of an algebra A , we see that not only commutativity of elements, but also associativity is implied. Specifically, we say that m is left commutative if the diagram

$$\begin{array}{ccccc} A \times A \times A & \xrightarrow{Id_A \times m} & A \times A & \xrightarrow{m} & A \\ T \times Id_A \downarrow & & & & \parallel \\ A \times A \times A & \xrightarrow{Id_A \times m} & A \times A & \xrightarrow{m} & A. \end{array}$$

commutes. This is a stronger requirement than commutativity of elements, but implies it since $a \cdot b = a \cdot (b \cdot 1) = b \cdot (a \cdot 1) = b \cdot a$ for all $a, b \in A$. It also implies associativity (cf. [LL]). A comultiplication operator Δ of a coalgebra C works similarly: these operators are (right) cocommutative, if the diagram

$$\begin{array}{ccccc}
A & \xrightarrow{\Delta} & A \times A & \xrightarrow{Id_A \times \Delta} & A \times A \times A \\
\parallel & & & & \downarrow T \times Id_A \\
A & \xrightarrow{\Delta} & A \times A & \xrightarrow{Id_A \times \Delta} & A \times A \times A.
\end{array}$$

commutes. This property, along with a counit c , yields both cocommutativity:

$$\Delta = (Id_A \otimes Id_A \otimes c)(Id_A \otimes \Delta)\Delta = (T \otimes c)(Id_A \otimes \Delta)\Delta = T\Delta,$$

and coassociativity:

$$\begin{aligned}
(Id_A \otimes \Delta)\Delta &= (Id_A \otimes T)(Id_A \otimes \Delta)\Delta \\
&= (Id_A \otimes T)(T \otimes Id_A)(Id_A \otimes \Delta)\Delta \\
&= (\Delta \otimes Id_A)T\Delta \\
&= (\Delta \otimes Id_A)\Delta
\end{aligned}$$

where the first and last equalities use cocommutativity of Δ .

Hence we focus our discussion on cocommutativity of comultiplication operators. Cocommutativity of comultiplication operators does not hold in all generality for the graded vertex coalgebras, however, the following weaker result holds. (In Proposition 3.2 we will see the extent to which this property implies coassociativity.)

Proposition 2.6. (weak cocommutativity) *Let V be a graded vertex algebra and $v' \in (V^{\otimes 3})'$. There exists $k \in \mathbb{Z}_+$ such that*

$$(2.23) \quad (x_1 - x_2)^k \langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v - (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2)v \rangle = 0$$

for any $v \in V$.

Proof. Multiplying the Jacobi identity by x_0^k and taking Res_{x_0} , we have

$$\begin{aligned}
(2.24) \quad & (x_1 - x_2)^k (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1) - (-x_2 + x_1)^k (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2) \\
&= Res_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) x_0^k (\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2).
\end{aligned}$$

Let $v' \in (V^{\otimes 3})'$, which is a (possibly infinitely) sum of elements the form $v'_1 \otimes v'_2 \otimes v'_3$ for $v'_1 \in V_{(r)}^*$, $v'_2 \in V_{(s)}^*$, $v'_3 \in V_{(t)}^*$, with $r + s + t = p$. Also pick N such that $V_{(n)} = 0$ for all $n \leq N$. We know that for some $M \in \mathbb{Z}_+$

$$(2.25) \quad \langle v'_1 \otimes v'_2, \mathcal{A}(x_0)v \rangle \in \mathbb{F}x_0^{-M}[[x_0]]$$

for any choice of $v \in V_{(q)}$, since a non-zero coefficient implies that $\text{wt } v'_1 + \text{wt } v'_2 = \text{wt } \Delta_n(v) = \text{wt } v + n + 1$ or $r + s = q + n + 1$, which implies that $r + s > N + n + 1$ or $-n - 1 > N - r - s$. Simply let $M \geq -N + r + s$. Again, a non-zero coefficient implies that $r + s = p - t > p - N$ so for all summands $v'_1 \otimes v'_2$ in v' , (2.25) holds provided $M \geq -N + r + s \geq p - 2N$.

Because the choice of M is independent of the vector $\mathcal{A}(x_0)$ acts on, for all $k > M$ we have

$$\text{Res}_{x_0} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) x_0^k \langle v', (\mathcal{A}(x_0) \otimes \text{Id}_V) \mathcal{A}(x_2) v \rangle = 0$$

for any choice of $v \in V$. Hence (2.24) implies the proposition. \square

We can understand a great deal about compositions of \mathcal{A} operators by investigating how to obtain them from rational functions. Our discussion parallels the discourse in [LL] on expansions into formal Laurent series. Let $\mathbb{F}[x_1, x_2]_S$ be the ring of rational functions obtained by inverting the products of (zero or more) elements of S , where S is the set of nonzero homogeneous linear polynomials in x_1 and x_2 . Let $\iota_{12} : \mathbb{F}[x_1, x_2]_S \rightarrow \mathbb{F}[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$ be defined by mapping an element $\frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t}$ to $\frac{g(x_1, x_2)}{x_1^r x_2^s}$ times $\frac{1}{(x_1 - x_2)^t}$ expanded in nonnegative powers of x_2 where $g(x_1, x_2) \in \mathbb{F}[x_1, x_2]$ and $r, s, t \in \mathbb{N}$. The map ι_{12} is injective and hence can be inverted on $\iota_{12}[\mathbb{F}[x_1, x_2]_S]$. This generalizes to finitely many formal variables, i.e. $\iota_{1 \dots n}$, and any ordering of the formal variables (cf. Section 3.1 of [FHL]).

Proposition 2.7. (a) **(right rationality)** *Let V be a graded vertex algebra, $v' \in (V^{\otimes 3})'$ and $v \in V$. Then the formal series $\langle v', (\text{Id}_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_1) v \rangle$ is in $\mathbb{F}[x_1, x_2^{-1}][[x_1^{-1}, x_2]]$ and in fact*

$$(2.26) \quad \langle v', (\text{Id}_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_1) v \rangle = \iota_{12} f(x_1, x_2)$$

where $f(x_1, x_2) \in \mathbb{F}[x_1, x_2]_S$ is uniquely determined and is of the form

$$(2.27) \quad f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t}$$

for some $g(x_1, x_2) \in \mathbb{F}[x_1, x_2]$ and $r, s, t \in \mathbb{N}$.

(b) **(cocommutativity)** *It is also the case that*

$$\iota_{12}^{-1} \langle v', (\text{Id}_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_1) v \rangle = \iota_{21}^{-1} \langle v', (T \otimes \text{Id}_V) (\text{Id}_V \otimes \mathcal{A}(x_1)) \mathcal{A}(x_2) v \rangle.$$

Proof. Let $t \in \mathbb{Z}_+$ be chosen relative to v' so that weak cocommutativity holds, i.e.

$$(2.28) \quad (x_1 - x_2)^t \langle v', (\text{Id}_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_1) v \rangle \\ = (x_1 - x_2)^t \langle v', (T \otimes \text{Id}_V) (\text{Id}_V \otimes \mathcal{A}(x_1)) \mathcal{A}(x_2) v \rangle.$$

This is an equality of formal power series in $\mathbb{F}[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$, but the left-hand side has only finitely many positive powers of x_1 by truncation (2.10) and only finitely many negative powers of x_2 by (2.25). Similarly the right-hand side of (2.28) has only finitely many positive powers of x_2 and finitely many negative powers of x_1 . Hence both sides of (2.28) must be equal to a Laurent polynomial (unique for a given t)

$$\frac{g(x_1, x_2)}{x_1^r x_2^s}$$

for some $g(x_1, x_2) \in \mathbb{F}[x_1, x_2]$, with $r, s \in \mathbb{Z}$. Let $f(x_1, x_2)$ be the rational function uniquely defined by

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^t x_2^s (x_1 - x_2)^t}.$$

Since neither $(x_1 - x_2)^t$ nor $\langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v \rangle$ have infinitely many negative powers of x_2 , we may multiply by $(x_1 - x_2)^{-t} \in \mathbb{F}[[x_1^{-1}, x_2]]$ so that

$$\begin{aligned} \langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v \rangle &= (x_1 - x_2)^{-t} (x_1 - x_2)^t \langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v \rangle \\ &= \iota_{12} f(x_1, x_2). \end{aligned}$$

(See Section 2.1 on [LL] for a discussion of when such products are defined.) Thus part (a) is satisfied. Similarly, using finite negative powers of x_1 of the right-hand side of (2.28), we observe that

$$\langle v', (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2)v \rangle = \iota_{21} f(x_1, x_2)$$

satisfying part (b). \square

Remark 2.8. *The proof of Proposition 2.7 actually demonstrates that right rationality and cocommutativity follow from the axioms of a graded vertex coalgebra without the Jacobi identity, but with the addition of weak cocommutativity (2.23).*

2.5. Coassociativity properties. In the same way as we have investigated cocommutativity, we now investigate coassociativity and achieve the following results.

Proposition 2.9. (weak coassociativity) *Let V be a graded vertex algebra and $v' \in (V^{\otimes 3})'$. There exists $k \in \mathbb{Z}_+$ such that*

$$(2.29) \quad (x_0 + x_2)^k \langle v', (\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2)v - (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_0 + x_2)v \rangle = 0$$

for any $v \in V$.

Proof. To achieve weak cocommutativity we took an appropriate residue of x_0 from the Jacobi identity and noted that the right-hand side had only finitely many negative x_0 terms. We will now use the same approach with x_1 and the second term of the Jacobi identity. Multiplying the Jacobi identity by x_1^k and taking Res_{x_1} , we have

$$(2.30) \quad (x_0 + x_2)^k (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_0 + x_2) - Res_{x_1} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \\ x_1^k (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2) = (x_2 + x_0)^k (\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2)$$

by applying (2.3) and (2.2) to the first term, and (2.3) to the last term.

Let $v' \in (V_{(p)}^{\otimes 3})'$, again a possibly infinitely sum of elements of the form $v'_1 \otimes v'_2 \otimes v'_3$ for $v'_1 \in V_{(r)}^*$, $v'_2 \in V_{(s)}^*$, $v'_3 \in V_{(t)}^*$. We consider elements of this form and again pick N such that $V_{(n)} = 0$ for all $n \leq N$. We know that for $M \geq p - 2N$

$$(2.31) \quad \langle v'_1 \otimes v'_3, \mathcal{A}(x_1)v \rangle \in \mathbb{F}x_1^{-M}[[x_1]]$$

for any choice of $v \in V$, as in (2.25). Thus for $k > M$ we see that

$$Res_{x_1} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) x_1^k \langle v'_1 \otimes v'_2 \otimes v'_3, (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2)v \rangle = 0$$

for any choice of $v \in V$, so (2.30) implies the proposition. \square

Again mirroring our above discussion, we investigate an interpretation of coassociativity in terms of rational functions.

Proposition 2.10. (a) **(left rationality)** *Let V be a graded vertex algebra, $v' \in (V^{\otimes 3})'$ and $v \in V$. Then the formal series $\langle v', (\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2)v \rangle$ is in $\mathbb{F}[x_0^{-1}, x_2][[x_0, x_2^{-1}]]$ and in fact*

$$\langle v', (\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2)v \rangle = \iota_{20} k(x_0, x_2)$$

where $k(x_0, x_2) \in \mathbb{F}[x_0, x_2]_S$ is uniquely determined and is of the form

$$k(x_0, x_2) = \frac{h(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t}$$

for some $h(x_0, x_2) \in \mathbb{F}[x_0, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) **(coassociativity)** *It is also the case that*

$$\iota_{12}^{-1} \langle v', (Id_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_1)v \rangle = \left(\iota_{20}^{-1} \langle v', (\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2)v \rangle \right) \Big|_{x_0=x_1-x_2}.$$

Proof. Let $t \in \mathbb{Z}_+$ be chosen relative to v' so that weak coassociativity holds, i.e.

(2.32)

$$(x_0 + x_2)^t \langle v', (\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2)v \rangle = (x_0 + x_2)^t \langle v', (Id_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_0 + x_2)v \rangle.$$

The left-hand side of (2.32) has only finitely many positive powers of x_2 by truncation (2.10) and only finitely many negative powers of x_0 by (2.31). The right-hand side of (2.32), on the other hand, has only finitely many positive powers of x_0 and finitely many negative powers of x_2 . Hence both sides of (2.32) must be equal to a Laurent polynomial (unique given t)

$$\frac{h(x_0, x_2)}{x_0^r x_2^s}$$

for some $h(x_0, x_2) \in \mathbb{F}[x_0, x_2]$, with $r, s \in \mathbb{Z}$. Let $k(x_0, x_2)$ be the rational function defined by

$$k(x_0, x_2) = \frac{h(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t}.$$

Since neither $(x_0 + x_2)^t$ nor $\langle v', (\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2)v \rangle$ have infinitely many negative powers of x_0 , we may multiply by $(x_2 + x_0)^{-t} \in \mathbb{F}[[x_2^{-1}, x_0]]$ so that

$$\begin{aligned} \langle v', (\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2)v \rangle &= (x_2 + x_0)^{-t} (x_0 + x_2)^t \langle v', (\mathcal{A}(x_0) \otimes Id_V) \mathcal{A}(x_2)v \rangle \\ &= \iota_{20} k(x_0, x_2). \end{aligned}$$

This satisfies part (a). Since the right-hand side of (2.32) has only finite negative powers of x_2 , we may similarly verify that

$$\langle v', (Id_V \otimes \mathcal{A}(x_2)) \mathcal{A}(x_0 + x_2)v \rangle = \iota_{02} k(x_0, x_2).$$

Thus

$$\begin{aligned}
\iota_{12}^{-1}\langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v \rangle &= \iota_{02}^{-1}\langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_0 + x_2)v \rangle \Big|_{x_0=x_1-x_2} \\
&= k(x_0, x_2) \Big|_{x_0=x_1-x_2} \\
&= (\iota_{20}^{-1}\langle v', (\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2)v \rangle) \Big|_{x_0=x_1-x_2}.
\end{aligned}$$

□

Remark 2.11. *The proof of Proposition 2.10 actually demonstrates that left rationality and coassociativity follow from the axioms of a graded vertex coalgebra without the Jacobi identity, but with the addition of weak coassociativity (2.29).*

3. EQUIVALENT CHARACTERIZATIONS OF GRADED VERTEX COALGEBRAS

As we saw at the beginning of Section 2.4, in classical algebra cocommutativity of comultiplication operators implies cocommutativity *and* coassociativity. An analog of this result is true for graded vertex coalgebras. We use these observations to investigate a way to replace the Jacobi identity in the definition of graded vertex coalgebra.

Although, there is good reason to keep the Jacobi identity in the definition of vertex algebra or coalgebra (for instance that of keeping the definitions of a vertex (co)algebra and a (co)module over a vertex (co)algebra in the same form), it is useful to have equivalent characterizations of the Jacobi identity. In this section we investigate two equivalent characterizations of the Jacobi identity, and hence alternate characterizations of graded vertex coalgebras in general. Note that these characterizations do not apply to vertex coalgebras in general because (2.25) and (2.31) arise from the grading.

3.1. The Jacobi identity from cocommutativity and the D^* -bracket. We begin by claiming Proposition 7.4.1 from [Hub1]. The statement there is about the Jacobi identity being replaced in the context of a vertex operator coalgebra, but the proof is void of any reference to conformal structure (i.e. a representation of the Virasoro algebra) and is applicable to graded vertex coalgebras without modification.

Proposition 3.1. *The Jacobi identity may be replaced in the definition of a graded vertex algebra by right and left rationality, cocommutativity and coassociativity, i.e. the claims of Propositions 2.7 and 2.10.*

Putting together several of the preceding results, it is now possible to argue that the Jacobi identity is a consequence of weak cocommutativity and the D^* -bracket.

Proposition 3.2. *In the presence of all the axioms of a graded vertex coalgebra except the Jacobi identity, weak cocommutativity (2.23) and the D^* -bracket (2.20) imply the Jacobi identity.*

Proof. First, we will show that Equation (2.17) and skew-symmetry (2.18) hold without use of the Jacobi identity, and then show that weak coassociativity also holds. Via Remarks 2.8 and 2.11, that means that right and left rationality, cocommutativity and coassociativity all hold. Thus, by Proposition 3.1, the Jacobi identity holds.

Since the D^* -bracket immediately implies (2.22), by (2.15) we have

$$\begin{aligned} (Id_V \otimes c)\mathcal{A}(x+x_0) &= (Id_V \otimes ce^{-x_0D^*})\mathcal{A}(x)e^{x_0D^*} \\ &= (Id_V \otimes c)\mathcal{A}(x)e^{x_0D^*}. \end{aligned}$$

By the cocreation axiom, we may set $x = 0$ and get (2.17), i.e.

$$(3.1) \quad (Id_V \otimes c)\mathcal{A}(x_0) = e^{x_0D^*}.$$

Now let $v' \in (V^{\otimes 2})'$ and choose $k \in \mathbb{Z}_+$ so that weak cocommutativity holds on $v' \otimes c$. Applying (3.1), then (2.22) to the right-hand side of weak commutativity, we have

$$\begin{aligned} (x_1 - x_2)^k \langle v' \otimes c, (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v \rangle \\ &= (x_1 - x_2)^k \langle v' \otimes c, (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2)v \rangle \\ &= (x_1 - x_2)^k \langle v', T(Id_V \otimes e^{x_1D^*})\mathcal{A}(x_2)v \rangle \\ &= (x_1 - x_2)^k \langle v', T\mathcal{A}(x_2 - x_1)e^{x_1D^*}v \rangle. \end{aligned}$$

But since $(Id_V \otimes c)\mathcal{A}(x_2) \in V[[x_2]]$, we may take $x_2 = 0$ and get

$$x_1^k \langle v', \mathcal{A}(x_1)v \rangle = x_1^k \langle v', T\mathcal{A}(-x_1)e^{x_1D^*}v \rangle.$$

Multiplying by x_1^{-k} is well-defined so we get

$$(3.2) \quad \langle v', \mathcal{A}(x_1)v \rangle = \langle v', T\mathcal{A}(-x_1)e^{x_1D^*}v \rangle,$$

a form of skew-symmetry (independent of v' and v).

Finally, let $v' \in (V^{\otimes 3})'$ and choose k such that weak commutativity holds for $v'(Id_V \otimes T)$. Employing (3.2), (2.22), weak cocommutativity, and (3.2), in order, we have

$$\begin{aligned} (x_0 + x_2)^k \langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_0 + x_2)v \rangle \\ &= (x_0 + x_2)^k \langle v', (Id_V \otimes T\mathcal{A}(-x_2)e^{x_2D^*})\mathcal{A}(x_0 + x_2)v \rangle \\ &= (x_0 + x_2)^k \langle v', (Id_V \otimes T\mathcal{A}(-x_2))\mathcal{A}(x_0)e^{x_2D^*}v \rangle \\ &= (x_0 + x_2)^k \langle v', (Id_V \otimes T)(T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_0))\mathcal{A}(-x_2)e^{x_2D^*}v \rangle \\ &= (x_0 + x_2)^k \langle v', (Id_V \otimes T)(T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_0))T\mathcal{A}(x_2)v \rangle \\ &= (x_0 + x_2)^k \langle v', (\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2)v \rangle. \end{aligned}$$

This is weak coassociativity, completing our proof. \square

Proposition 3.2 allows us to give the following equivalent definition of a graded vertex coalgebra.

Definition 3.3. *A graded vertex coalgebra may be described as a collection of data*

- a \mathbb{Z} -graded vector space $V = \coprod_{k=-N}^{\infty} V_{(k)}$, with $N \in \mathbb{N}$
- a linear map $c : V \mapsto \mathbb{F}$

- a linear map

$$\begin{aligned} \mathcal{A}(x) : V &\rightarrow (V \otimes V)((x^{-1})) \\ v &\mapsto \mathcal{A}(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v)x^{-k-1}, \end{aligned}$$

such that for $v \in V_{(j)}$, $\Delta_k(v) \in (V \otimes V)_{(j+k+1)}$,

satisfying the following axioms for $v \in V$ and $v' \in (V^{\otimes 3})'$:

1. $(c \otimes Id_V)\mathcal{A}(x) = Id_V$
2. $(Id_V \otimes c)\mathcal{A}(x)v \in V[[x]]$, and $\lim_{x \rightarrow 0} (Id_V \otimes c)\mathcal{A}(x)v = v$
3. $\frac{d}{dx}\mathcal{A}(x) = \mathcal{A}(x)D^* - (Id_V \otimes D^*)\mathcal{A}(x)$ where $D^* = (Id_V \otimes c)\Delta_{-2}$
4. There exists $k \in \mathbb{Z}_+$ depending on v' but independent of v such that

$$(x_1 - x_2)^k \langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v - (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2)v \rangle = 0.$$

Here, we combine the truncation axiom into the definition of the range of $\mathcal{A}(x)$.

3.2. The Jacobi identity from coassociativity and skew-symmetry. There is another equivalent substitute for the Jacobi identity besides weak cocommutativity and the D^* -bracket, which also yields an equivalent characterization of graded vertex coalgebra.

Proposition 3.4. *In the presence of all the axioms of a graded vertex coalgebra except the Jacobi identity, weak coassociativity (2.29) and skew-symmetry (2.18) imply the Jacobi identity.*

Proof. Mirroring the proof of Proposition 3.2, we need only prove that weak co-commutativity is satisfied. First, note that the left counit axiom (2.7) and skew-symmetry (2.18) imply (2.17). Now let $v' \in (V^{\otimes 2})'$ and choose $k \in \mathbb{Z}_+$ so that weak coassociativity holds on $v' \otimes c$:

$$(x_0 + x_2)^k \langle v' \otimes c, (\mathcal{A}(x_0) \otimes Id_V)\mathcal{A}(x_2)v \rangle = (x_0 + x_2)^k \langle v' \otimes c, (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_0 + x_2)v \rangle.$$

Adding (2.17), we get

$$(x_0 + x_2)^k \langle v', \mathcal{A}(x_0)e^{x_2 D^*}v \rangle = (x_0 + x_2)^k \langle v', (Id_V \otimes e^{x_2 D^*})\mathcal{A}(x_0 + x_2)v \rangle.$$

Since $(x_0 + x_2)^k$, $\langle v', \mathcal{A}(x_0)e^{x_2 D^*}v \rangle$ and $\langle v', (Id_V \otimes e^{x_2 D^*})\mathcal{A}(x_0 + x_2)v \rangle$ all involve no negative powers of x_2 , we may multiply both sides by $(x_0 + x_2)^{-k}$ to get

$$\langle v', \mathcal{A}(x_0)e^{x_2 D^*}v \rangle = \langle v', (Id_V \otimes e^{x_2 D^*})\mathcal{A}(x_0 + x_2)v \rangle$$

which is independent of v' and v , or equivalently,

$$(3.3) \quad (Id_V \otimes e^{-x_2 D^*})\mathcal{A}(x_0) = \mathcal{A}(x_0 + x_2)e^{-x_2 D^*}.$$

Finally, let $v' \in (V^{\otimes 3})'$, pick $k \in \mathbb{Z}_+$ so that weak coassociativity holds for $v'(T \otimes Id_V)(Id_V \otimes T)$, i.e. for any $v \in V$

$$(3.4) \quad (x_2 - x_1)^k \langle v', (T \otimes Id_V)(Id_V \otimes T)(\mathcal{A}(x_2) \otimes Id_V)\mathcal{A}(-x_1)v \rangle \\ = (x_2 - x_1)^k \langle v', (T \otimes Id_V)(Id_V \otimes T)(Id_V \otimes \mathcal{A}(-x_1))\mathcal{A}(x_2 - x_1)v \rangle.$$

Successively using skew-symmetry, (3.4), (3.3), and skew-symmetry, we have

$$(x_2 - x_1)^k \langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v \rangle \\ = (x_2 - x_1)^k \langle v', (T \otimes Id_V)(Id_V \otimes T)(\mathcal{A}(x_2) \otimes Id_V)\mathcal{A}(-x_1)e^{x_1 D^*} v \rangle \\ = (x_2 - x_1)^k \langle v', (T \otimes Id_V)(Id_V \otimes T)(Id_V \otimes \mathcal{A}(-x_1))\mathcal{A}(x_2 - x_1)e^{x_1 D^*} v \rangle \\ = (x_2 - x_1)^k \langle v', (T \otimes Id_V)(Id_V \otimes T)(Id_V \otimes \mathcal{A}(-x_1)e^{x_1 D^*})\mathcal{A}(x_2)v \rangle \\ = (x_2 - x_1)^k \langle v', (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2)v \rangle.$$

This is a form of weak cocommutativity. \square

The proposition allows us to consider weak coassociativity and skew-symmetry as a replacement for the Jacobi identity in the definition of graded vertex coalgebra.

4. COMODULES

We now study comodules over vertex coalgebras. Intuitively, we desire to investigate a vector space under which left comultiplication by a given vertex coalgebra satisfies all of the axioms of vertex coalgebra that make sense.

Definition 4.1. *A comodule M for a vertex coalgebra (V, \mathcal{A}_V, c) is a vector space equipped with a linear map*

$$\mathcal{A}_M(x) : M \mapsto (V \otimes M)[[x, x^{-1}]]$$

such that for all $w \in M$,

1. *Left counit:* $(c \otimes Id_M)\mathcal{A}_M(x)w = w$
2. *Truncation:* $\mathcal{A}_M(x)w \in (V \otimes M)((x^{-1}))$
3. *Jacobi Identity:*

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) (Id_V \otimes \mathcal{A}_M(x_2))\mathcal{A}_M(x_1) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) (T \otimes Id_M) \\ (Id_V \otimes \mathcal{A}_M(x_1))\mathcal{A}_M(x_2) = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) (\mathcal{A}_V(x_0) \otimes Id_M)\mathcal{A}_M(x_2).$$

Moving to graded objects, we follow a precedent in vertex (operator) algebras and allow comodules over graded vertex coalgebras to be \mathbb{Q} -graded (cf. [FHL], although \mathbb{C} -grading is sometimes used).

Definition 4.2. *A \mathbb{Q} -graded vector space $M = \coprod_{j \in \mathbb{Q}} M_{(j)}$, such that $M_{(j)} = 0$ for $j \ll 0$, is considered a comodule over a graded vertex coalgebra V if it is a comodule over V as an ungraded vertex coalgebra such that the map $\mathcal{A}_M(x) : M \mapsto (V \otimes M)[[x, x^{-1}]]$ may be described by*

$$w \mapsto \mathcal{A}_M(x)w = \sum_{k \in \mathbb{Z}} \Delta_k(w)x^{-k-1}$$

where for $w \in M_{(j)}$, $\Delta_k(w) \in (V \otimes M)_{(j+k+1)}$. The graded vertex coalgebra will be said to be finite dimensionally graded if $\dim M_{(j)} < \infty$ for each $j \in \mathbb{Q}$.

Much of the work of studying graded vertex coalgebras extends to comodules of graded vertex coalgebras. Referring to the proofs of Propositions 2.6, 2.7, 2.9 and 2.10 immediately give us the following propositions for comodules.

Proposition 4.3. (weak cocommutativity) *Given a comodule M over a graded vertex coalgebra V , let $v' \in (V \otimes V \otimes M)'$. There exists $k \in \mathbb{Z}_+$ such that*

$$(4.1) \quad (x_1 - x_2)^k \langle v', (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_1) w - (T \otimes Id_M)(Id_V \otimes \mathcal{A}_M(x_1)) \mathcal{A}_M(x_2) w \rangle = 0$$

for any $w \in M$.

In particular, we have the analog of Equation (2.25) for comodules. For $v' \in V'$, $w' \in M'$, there exists some $K \in \mathbb{Z}_+$

$$(4.2) \quad \langle v' \otimes w', \mathcal{A}_M(x_0) w \rangle \in \mathbb{F} x_0^{-K} [[x_0]]$$

for any choice of $w \in M$.

Proposition 4.4. (a) (right rationality) *Let a vector space M satisfy all the axioms of being a comodule over a graded vertex coalgebra V except the Jacobi identity, but also satisfy weak cocommutativity. Also let $v' \in (V \otimes V \otimes M)'$ and $w \in M$. Then the formal series $\langle v', (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_1) w \rangle$ is in $\mathbb{F}[x_1, x_2^{-1}][[x_1^{-1}, x_2]]$ and in fact*

$$\langle v', (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_1) w \rangle = \iota_{12} f(x_1, x_2)$$

where $f(x_1, x_2) \in \mathbb{F}[x_1, x_2]_S$ is uniquely determined and is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t}$$

for some $g(x_1, x_2) \in \mathbb{F}[x_1, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) **(cocommutativity)** *It is also the case that*

$$(4.3) \quad \iota_{12}^{-1} \langle v', (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_1) w \rangle = \iota_{21}^{-1} \langle v', (T \otimes Id_M)(Id_V \otimes \mathcal{A}_M(x_1)) \mathcal{A}_M(x_2) w \rangle.$$

Proposition 4.5. (weak coassociativity) *Given a comodule M over a graded vertex coalgebra V , let $v' \in (V \otimes V \otimes M)'$. There exists $k \in \mathbb{Z}_+$ such that*

$$(4.4) \quad (x_0 + x_2)^k \langle v', (\mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_2) w - (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_0 + x_2) w \rangle = 0$$

for any $w \in M$.

Proposition 4.6. (a) (left rationality) *Let a vector space M satisfy all the axioms of being a comodule over a graded vertex coalgebra V except the Jacobi identity, but also satisfy weak coassociativity. Also let $v' \in (V \otimes V \otimes M)'$ and $w \in M$. Then the formal series $\langle v', (\mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_2) w \rangle$ is in $\mathbb{F}[x_0^{-1}, x_2][[x_0, x_2^{-1}]]$ and in fact*

$$\langle v', (\mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_2) w \rangle = \iota_{20} k(x_0, x_2)$$

where $k(x_0, x_2) \in \mathbb{F}[x_0, x_2]_S$ is uniquely determined and is of the form

$$(4.5) \quad k(x_0, x_2) = \frac{h(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t}$$

for some $h(x_0, x_2) \in \mathbb{F}[x_0, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) **(coassociativity)** It is also the case that

$$(4.6) \quad \iota_{12}^{-1} \langle v', (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_1) w \rangle = \left(\iota_{20}^{-1} \langle v', (\mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_2) w \rangle \right) \Big|_{x_0=x_1-x_2}.$$

The trivial modifications that we have made to right and left rationality, cocommutativity and coassociativity allow us also to claim Proposition 3.1 in the graded vertex coalgebra comodule context.

Proposition 4.7. *The Jacobi identity may be replaced in the definition of a comodule over graded vertex coalgebra by right and left rationality, cocommutativity and coassociativity, i.e. the claims of Propositions 4.4 and 4.6.*

Unlike the case in graded vertex coalgebras, where the most natural substitute for the Jacobi identity would seem to be weak cocommutativity and the D^* -bracket, for comodules of graded vertex coalgebras we are able to use only weak coassociativity to obtain the Jacobi identity. (The D^* -bracket formula does not make sense in the comodule context, and it is not known if cocommutativity itself implies the Jacobi identity.)

Proposition 4.8. *Let a vector space M satisfy all the axioms of being a comodule over a graded vertex coalgebra V except the Jacobi identity, but also satisfy weak coassociativity (4.4). Then M satisfies the Jacobi identity.*

Proof. By Proposition 4.6, we have left rationality and coassociativity. We need only show weak coassociativity of comodules implies right rationality and cocommutativity, for then Proposition 4.7 implies the Jacobi identity. Notice that right rationality is actually a direct consequence of coassociativity (4.6) so our main task is to show cocommutativity (4.3).

We start by showing that the D^* -derivative holds for comodule comultiplication. Let $v' \in V'$, $w' \in M'$ and pick $k \in \mathbb{Z}_+$ so that weak coassociativity holds for $v' \otimes c \otimes w'$. So for all $w \in M$,

$$\begin{aligned} (x_0 + x_2)^k \langle v' \otimes c \otimes w', (\mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_2) w \rangle \\ = (x_0 + x_2)^k \langle v' \otimes c \otimes w', (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_0 + x_2) w \rangle. \end{aligned}$$

Applying (2.17) on the left-hand side and the left counit property on the right, we get

$$(x_0 + x_2)^k \langle v' \otimes w', (e^{x_0 D^*} \otimes Id_M) \mathcal{A}_M(x_2) w \rangle = (x_0 + x_2)^k \langle v' \otimes w', \mathcal{A}_M(x_0 + x_2) w \rangle.$$

Recalling (4.2), we may choose k larger if necessary so that $x^k \langle v' \otimes w', \mathcal{A}_M(x) w \rangle \in \mathbb{F}[[x]]$. For positive powers, $x_0 + x_2$ and $x_2 + x_0$ are equivalent, so we may write

$$(x_0 + x_2)^k \langle v' \otimes w', (e^{x_0 D^*} \otimes Id_V) \mathcal{A}_M(x_2) w \rangle = (x_0 + x_2)^k \langle v' \otimes w', \mathcal{A}_M(x_2 + x_0) w \rangle.$$

Finally, multiply both sides by $(x_0 + x_2)^{-k}$ and observe that independent of v' , w' , w ,

$$(4.7) \quad (e^{x_0 D^*} \otimes Id_M) \mathcal{A}_M(x_2) = \mathcal{A}_M(x_2 + x_0).$$

Now by left rationality (a consequence of weak coassociativity), for $v' \in (V \otimes V \otimes M)'$, $w \in M$ we have a rational function of the form

$$k(x_0, x_2) = \frac{h(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t},$$

with $h(x_0, x_2) \in \mathbb{F}[x_0, x_2]$, $r, s, t \in \mathbb{Z}$ such that

$$(4.8) \quad \langle v', (\mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_2) w \rangle = \iota_{20} k(x_0, x_2).$$

We know that by coassociativity, the left-hand side of the cocommutativity equation (4.3) satisfies

$$\begin{aligned} \iota_{12}^{-1} \langle v', (Id_V \otimes \mathcal{A}_M(x_2)) \mathcal{A}_M(x_1) v \rangle &= (\iota_{20}^{-1} \langle v', (\mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_2) w \rangle) \Big|_{x_0=x_1-x_2} \\ &= k(x_0, x_2) \Big|_{x_0=x_1-x_2}. \end{aligned}$$

Now working from the right-hand side of cocommutativity (4.3) and using coassociativity, (2.18), (4.7), and (4.8) we also have

$$\begin{aligned} \iota_{21}^{-1} \langle v', (T \otimes Id_V) (Id_V \otimes \mathcal{A}_M(x_1)) \mathcal{A}_M(x_2) v \rangle &= (\iota_{10}^{-1} \langle v', (T \mathcal{A}_V(x_0) \otimes Id_M) \mathcal{A}_M(x_1) w \rangle) \Big|_{x_0=x_2-x_1} \\ &= (\iota_{10}^{-1} \langle v', (\mathcal{A}_V(-x_0) e^{x_0 D^*} \otimes Id_M) \mathcal{A}_M(x_1) w \rangle) \Big|_{x_0=x_2-x_1} \\ &= (\iota_{10}^{-1} \langle v', (\mathcal{A}_V(-x_0) \otimes Id_M) \mathcal{A}_M(x_1 + x_0) w \rangle) \Big|_{x_0=x_2-x_1} \\ &= k(-x_0, x_1 + x_0) \Big|_{x_0=x_2-x_1}. \end{aligned}$$

But as rational functions,

$$k(x_0, x_2) \Big|_{x_0=x_1-x_2} = \frac{h(x_1 - x_2, x_2)}{(x_1 - x_2)^r x_2^s x_1^t} = k(-x_0, x_1 + x_0) \Big|_{x_0=x_2-x_1}.$$

This completes the proof of cocommutativity. \square

Hence, comodules of graded vertex coalgebras may now be axiomatized as follows.

Definition 4.9. *A comodule over a graded vertex coalgebra V may be described as a collection of data*

1. a \mathbb{Q} -graded vector space $V = \coprod_{k \in \mathbb{Q}} V_{(k)}$, with $V_{(k)} = 0$ for $k \ll 0$
2. a linear map

$$\begin{aligned} \mathcal{A}_M(x) : M &\rightarrow (V \otimes M)((x^{-1})) \\ v &\mapsto \mathcal{A}_M(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v) x^{-k-1}, \end{aligned}$$

such that for $v \in V_{(r)}$, $\Delta_k(v) \in (V \otimes M)_{(r+k+1)}$,

satisfying the following axioms for $v' \in (V \otimes V \otimes M)'$ and $w \in M$:

1. $(c \otimes Id_M)\mathcal{A}_M(x) = Id_M$
2. There exists $k \in \mathbb{Z}_+$ depending on v' but independent of w such that

$$(x_0 + x_2)^k \langle v', (\mathcal{A}_V(x_0) \otimes Id_M)\mathcal{A}_M(x_2)w - (Id_V \otimes \mathcal{A}_M(x_2))\mathcal{A}_M(x_0 + x_2)w \rangle = 0.$$

5. VERTEX OPERATOR COALGEBRAS AND THEIR COMODULES

Starting with [Hub1] and continuing in [Hub2], [Hub3] and [Hub4], a particular type of graded vertex algebra, called a vertex operator coalgebra, has been studied which possesses a conformal structure. We note that the cocommutativity, coassociativity and comodule results for graded vertex coalgebras extend automatically to vertex operator coalgebras.

A vertex operator coalgebra may be thought of as a vertex algebra with a natural action of the Virasoro algebra. Natural in this context means that the grading of the vertex operator algebra is the one determined by the generator $L(0)$ of the Virasoro algebra and $L(1)$ acts as the operator D^* .

Consequently, Propositions 2.3, 2.4, 2.6, 2.7, 2.9, 2.10, 3.2 and 3.4 all hold for vertex operator algebras. Not only do these propositions give insight into the structure of vertex operator coalgebras, they also justify the following equivalent definition.

Definition 5.1. *A vertex operator coalgebra of rank $d \in \mathbb{F}$ has the following data:*

- $V = \prod_{k=-N}^{\infty} V_{(k)}$, with $\dim V_{(k)} < \infty$, $N \in \mathbb{N}$
- a linear map $c : V \mapsto \mathbb{F}$,
- a linear map $\rho : V \mapsto \mathbb{F}$,
- a linear map

$$\begin{aligned} \mathcal{A}(x) : V &\mapsto (V \otimes V)((x^{-1})) \\ v &\mapsto \mathcal{A}(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v)x^{-k-1} \end{aligned}$$

satisfying (for $v \in V$):

1. *Left counit:* $(c \otimes Id_V)\mathcal{A}(x)v = v$
2. *Cocreation:* $(Id_V \otimes c)\mathcal{A}(x)v \in V[[x]]$
 $\lim_{x \rightarrow 0}(Id_V \otimes c)\mathcal{A}(x)v = v$
3. *Weak cocommutativity:* for $v' \in (V^{\otimes 3})'$, there exists $k \in \mathbb{Z}_+$ (depending on v' but not on v) such that

$$(x_1 - x_2)^k \langle v', (Id_V \otimes \mathcal{A}(x_2))\mathcal{A}(x_1)v - (T \otimes Id_V)(Id_V \otimes \mathcal{A}(x_1))\mathcal{A}(x_2)v \rangle = 0$$

4. *Virasoro algebra:* given $(\rho \otimes Id_V)\mathcal{A}(x) = \sum_{k \in \mathbb{Z}} L(k)x^{k-2}$, for all $j, k \in \mathbb{Z}$

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j, -k}d,$$

5. *Grading:* if $v \in V_{(k)}$, $L(0)v = kv$
6. *$L(1)$ -Derivative and bracket:*

$$\frac{d}{dx}\mathcal{A}(x) = (L(1) \otimes Id_V)\mathcal{A}(x) = \mathcal{A}(x)L(1) - (Id_V \otimes L(1))\mathcal{A}(x).$$

In practice weak cocommutativity is easier to verify than the Jacobi identity, making this description preferable in contexts where verification is necessary.

A comodule over a vertex operator coalgebra has a similar alternate definition as a result of Propositions 4.3, 4.4, 4.5, 4.6, and 4.8

Definition 5.2. *A vertex operator coalgebra comodule of rank $d \in \mathbb{F}$ over V has the following data:*

- $M = \coprod_{k \in \mathbb{Q}} M_{(k)}$, with $\dim M_{(k)} < \infty$ for all k , $M_{(k)} = 0$ for $k \ll 0$,
- a linear map

$$\begin{aligned} \mathcal{A}_M(x) : M &\mapsto (V \otimes M)((x^{-1})) \\ w &\mapsto \mathcal{A}_M(x)w = \sum_{k \in \mathbb{Z}} \Delta_k(w)x^{-k-1} \end{aligned}$$

satisfying (for $w \in M$):

1. *Left counit:* $(c \otimes Id_V)\mathcal{A}_M(x)w = w$
2. *Weak coassociativity:* for $v' \in (V \otimes V \otimes M)'$, there exists $k \in \mathbb{Z}_+$ depending on v' (but not on w) such that

$$(x_0 + x_2)^k \langle v', (\mathcal{A}_V(x_0) \otimes Id_M)\mathcal{A}_M(x_2)w - (Id_V \otimes \mathcal{A}_M(x_2))\mathcal{A}_M(x_0 + x_2)w \rangle = 0$$

3. *Virasoro algebra:* given $(\rho \otimes Id_M)\mathcal{A}_M(x) = \sum_{k \in \mathbb{Z}} L(k)x^{k-2}$, for all $j, k \in \mathbb{Z}$

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j, -k}d,$$

4. *Grading:* if $w \in M_{(k)}$, $L(0)w = kw$
5. *$L(1)$ -Derivative and bracket:*

$$\begin{aligned} \frac{d}{dx}\mathcal{A}_M(x) &= (L(1) \otimes Id_M)\mathcal{A}_M(x) \\ &= \mathcal{A}_M(x)L(1) - (Id_V \otimes L(1))\mathcal{A}_M(x). \end{aligned}$$

6. THE CATEGORIES OF FINITE DIMENSIONALLY GRADED VERTEX COALGEBRAS AND FINITE DIMENSIONALLY GRADED VERTEX ALGEBRAS

An additional benefit of studying finite dimensionally graded vertex coalgebras is that grading allows us to pass naturally between a vector space and its graded dual, then back again while the finite dimensionality prevents enlarging our vector space. First, we restrict our attention to graded vertex coalgebras as described above and to the category of graded vertex algebras, by which we mean a vertex algebra $(V, Y, \mathbf{1})$ such that $V = \coprod_{k \in \mathbb{Z}} V_{(k)}$ with all $V_{(k)}$ trivial for $k \ll 0$, and for each $u \in V_{(j)}$, $v \in V_{(k)}$, we have $u_n v \in V_{(j+k-n-1)}$.

Here we mainly follow the approach in [Hub4].

Theorem 6.1. *Let $V = \coprod_{k \in \mathbb{Z}} V_{(k)}$ be a \mathbb{Z} -graded vector space. Choose a distinguished vectors $\mathbf{1} \in V_{(0)}$ and a linear map*

$$Y(\cdot, x) : V \rightarrow (\text{End } V)[[x, x^{-1}]].$$

Additionally, define $c \in V''_{(0)}$ to be the double dual of $\mathbf{1}$, and a linear operator $\mathcal{A}(x) : V' \rightarrow (V' \otimes V')[[x, x^{-1}]]$ defined by

$$\langle \mathcal{A}(x)u', v \otimes w \rangle = \langle u', Y(v, x)w \rangle$$

for all $u' \in V'$, $v, w \in V$. Then $(V, Y, \mathbf{1})$ is a graded vertex algebra if and only if (V', \mathcal{A}, c) is a graded vertex coalgebra.

The proof follows that of Theorem 2.9 in [Hub4] quite neatly, but with two interesting features. First, the Virasoro bracket structure is not intermingled with the proof of any of the other axioms and thus may be entirely removed. Second, the gradation, or specifically the weight condition $u_n v \in V_{(j+k-n-1)}$ for $u \in V_{(j)}$, $v \in V_{(k)}$, is essential to proving the truncation condition. The proof of Theorem 6.1 depends on having *graded* vertex algebras and coalgebras.

As in [Hub4], Theorem 6.1 gives rise to a contravariant functor from the category of graded vertex algebras to the category of graded vertex coalgebras, and a similar functor goes in the opposite direction. When V is a finite dimensionally graded vertex algebra, V and V'' are canonically isomorphic, and these functors demonstrate an equivalence of categories.

Corollary 6.2. *The categories of finite dimensionally graded vertex algebras and finite dimensionally graded vertex coalgebras are equivalent.*

7. MOTIVATION FROM THE UNIVERSAL ENVELOPING ALGEBRA OF A FINITE DIMENSIONAL LIE ALGEBRA

Vertex algebras are distinctive in that they exhibit properties of associative algebras and of Lie algebras. In trying to determine the correct “bialgebraic” structures appearing in the vertex algebra world, it is productive to consider the most widely known example of bialgebras - the universal enveloping algebra of a Lie algebra. Given a finite dimensional abelian Lie algebra \mathcal{L} , we construct the universal enveloping algebra of \mathcal{L} by first considering the tensor algebra of \mathcal{L} as a vector space

$$T(\mathcal{L}) = \prod_{i \in \mathbb{N}} \mathcal{L}^{\otimes i}$$

where $\mathcal{L}^{\otimes i} = \underbrace{\mathcal{L} \otimes \cdots \otimes \mathcal{L}}_i$ (and $\mathcal{L}^{\otimes 0} = \mathbb{F}$). (We could use a Heisenberg algebra,

or any Lie algebra where the center contains the commutator, but an abelian Lie algebra will do for the purpose of the current analogy.) Let J be the ideal generated by the tensors

$$x \otimes y - y \otimes x - [x, y]$$

for $x, y \in \mathcal{L}$. The universal enveloping algebra is the associative, coassociative bialgebra

$$U(\mathcal{L}) = T(\mathcal{L})/J$$

where multiplication is given by

$$m(u, v) = u \otimes v$$

for $u, v \in U(\mathcal{L})$.

Now any coalgebra structure on a vectors space V naturally gives rise to an algebra structure on the dual space V^* , i.e. for $u', v' \in V^*$, $w \in V$, $m(u', v')(w) =$

$(u' \otimes v')(\Delta w)$. But an algebra structure on a vector space V only gives a coalgebra structure on a subspace of the dual,

$$V^\circ = \{f \in V^* \mid f(I) = 0 \text{ for some ideal } I \text{ of } V \text{ such that } \dim V/I < \infty\},$$

the so-called ‘finite dual’ (cf. [M]). The comultiplication $\Delta^* : V^\circ \rightarrow V^\circ \otimes V^\circ$ is determined by

$$\Delta^*(u')(v \otimes w) = u'(m(v, w))$$

for $u' \in V^\circ$, $v, w \in V$, where m is the multiplication on V .

In order for $U(\mathcal{L})$ to be a bialgebra however, we must have a comultiplication on the vector space itself rather than on the finite dual. We will construct the comultiplicative structure on $U(\mathcal{L})$ as follows. By the Poincaré-Birkhoff-Witt theorem, a basis e_1, \dots, e_n of \mathcal{L} determines a basis from $U(\mathcal{L})$, which is

$$\{e_{i_1} \otimes \dots \otimes e_{i_k} \mid k \in \mathbb{N}, i_1 \leq i_2 \leq \dots \leq i_k \leq n\}$$

We then have a linear map

$$\Phi : U(\mathcal{L}) \rightarrow U(\mathcal{L})^\circ$$

determined by mapping each basis element $e_{i_1} \otimes \dots \otimes e_{i_k}$ to the dual element $f \in U(\mathcal{L})^*$ which maps $e_{i_1} \otimes \dots \otimes e_{i_k}$ to one and all other basis elements to 0. Certainly $f \in U(\mathcal{L})^\circ$ since allowing

$$I = \text{span}\{e_{j_1} \otimes \dots \otimes e_{j_\ell} \mid \ell > k, j_1 \leq j_2 \leq \dots \leq j_\ell \leq n\},$$

we have $f(I) = 0$ and $\dim(U(\mathcal{L})/I) < \infty$. The map Φ is definitely injective, and surjectivity may be seen as follows. Any $f \in U(\mathcal{L})^*$ has a set of basis elements $e_{i_1} \otimes \dots \otimes e_{i_k}$ for which $f(e_{i_1} \otimes \dots \otimes e_{i_k}) \neq 0$. If the set is finite, $f \in \Phi[U(\mathcal{L})]$. If the set is infinite, it is impossible to pick an ideal I of $U(\mathcal{L})$ such that $f(I) = 0$ and $\dim U(\mathcal{L})/I < \infty$. Hence, Φ is a vector space isomorphism.

Using the isomorphism Φ , we induce a comultiplication on $U(\mathcal{L})$ from the comultiplication Δ^* on $U(\mathcal{L})^\circ$,

$$\Delta = (\Phi \otimes \Phi)^{-1} \circ \Delta^* \circ \Phi.$$

We will see that this comultiplication is the ‘classical’ (and unique, cf. [U]) comultiplication Δ on $U(\mathcal{L})$. We need only examine Δ on $\mathcal{L} \subset U(\mathcal{L})$, since the comultiplication is completely determined by these elements using the properties:

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(vw) = (m \otimes m) \circ (Id_{U(\mathcal{L})} \otimes T \otimes Id_{U(\mathcal{L})})(\Delta(v) \otimes \Delta(w)),$$

for any $v, w \in U(\mathcal{L})$. First, consider a basis element $u \in \mathcal{L} \subset U(\mathcal{L})$, and apply $\Delta^* \circ \Phi(u)$ to two arbitrary basis elements $v, w \in U(\mathcal{L})$. Denote by $u' \in U(\mathcal{L})^\circ$ the element that is 1 on u and zero on every other basis element.

$$\begin{aligned} \Delta^* \circ \Phi(u)(v \otimes w) &= \Delta^*(u')(v \otimes w) \\ &= u'(vw) \\ &= \begin{cases} 1, & \text{if } u = vw \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Thus $\Delta^* \circ \Phi(u)$ is the element of $U(\mathcal{L})^\circ \otimes U(\mathcal{L})^\circ$ that takes a value of 1 on the basis elements $u \otimes 1$ and $1 \otimes u$, but takes a value of zero on every other basis element. In other words, given u' the dual of u as above and $1'$ the dual of 1,

$$\Delta^* \circ \Phi(u) = u' \otimes 1' + 1' \otimes u'.$$

Therefore

$$\Delta(u) = u \otimes 1 + 1 \otimes u.$$

This is the classical description of the comultiplication of a universal enveloping algebra, but we have derived it by examining the comultiplication induced on the finite dual by the original *multiplication*.

With this as motivation, we now turn our attention the vertex algebras and repeat the process to find a ‘correct’ comultiplicative structure. Rather than posing this result in terms of dual vector spaces or finite duals, these results may be posed in terms of a nondegenerate bilinear form $(\cdot, \cdot) : V \otimes V \rightarrow \mathbb{F}$. Given any vector space V , a bilinear form on V determines a linear map from V to V^* , and vice versa. A nondegenerate bilinear form corresponds to an injective map $\Phi : V \rightarrow V^*$. Thus, our isomorphism $\Phi : U(\mathcal{L}) \rightarrow U(\mathcal{L})^\circ \subset U(\mathcal{L})^*$ determines a nondegenerate bilinear form on $U(\mathcal{L})$ and we may define Δ by

$$(\Delta(u), v \otimes w) = (u, m(v, w))$$

for all $u, v, w \in U(\mathcal{L})$.

We now employ the same technique on a graded vertex algebra $(V, Y, \mathbf{1})$. For each $k \in \mathbb{Z}$, choose a basis $\{e_{k,i}\}_{i \in I_k}$ for $V_{(k)}$. Then $\{e_{k,i} | k \in \mathbb{Z}, i \in I_k\}$ is a basis for V and determines a bilinear form $(\cdot, \cdot) : V \otimes V \rightarrow \mathbb{F}$ on V by the rule

$$(7.1) \quad (e_{k,i}, e_{\ell,j}) = \delta_{k,\ell} \delta_{i,j}.$$

In other words, the product of any two basis elements is 1 if they are the same and 0 if they are different. Hence the form is symmetric and nondegenerate by construction. It is also graded, i.e.,

$$(V_{(k)}, V_{(\ell)}) = 0$$

for $k \neq \ell$.

Theorem 7.1. *Given a graded vertex algebra $(V, Y, \mathbf{1})$ equipped with a graded, nondegenerate bilinear form, V also carries the structure of a vertex coalgebra with linear operators*

$$c : V \rightarrow \mathbb{F}, v \mapsto (v, \mathbf{1})$$

and

$$\mathcal{A}(x) : V \rightarrow (V \otimes V)[[x, x^{-1}]]$$

defined by $(\mathcal{A}(x)u, v \otimes w) = (u, Y(v, x)w)$ for $u, v, w \in V$.

The proof follows that of Theorem 3.1 in [Hub3] (which was stated in terms of vertex operator algebras and coalgebras) with weakened requirements. Instead of requiring a form that is invariant or Virasoro preserving, a graded form is all that is used in any portion of the proof besides the Virasoro bracket.

By the form constructed directly preceding Theorem 7.1, we conclude:

Corollary 7.2. *Every graded vertex algebra has a graded vertex coalgebra structure via the nondegenerate form determined in Equation (7.1).*

In the final section, we will extend the parallel with (associative) enveloping algebras by examining enveloping vertex algebras.

8. A GRADED VERTEX COALGEBRA STRUCTURE ON THE UNIVERSAL
ENVELOPING ALGEBRA OF VERTEX LIE ALGEBRAS

In [P], Primc defines the notion of vertex Lie algebra, which contains the “positive” information of a vertex algebra. He then constructs an enveloping vertex algebra for a vertex Lie algebra and goes on to show that the enveloping vertex algebra has the same universal property as that of an associative enveloping algebra. Namely, consider a (vertex) Lie Algebra \mathcal{L} , a (vertex) algebra A , and a (vertex) Lie algebra homomorphism $\phi : \mathcal{L} \rightarrow A$, viewing A as a (vertex) Lie algebra. There is a unique embedding $\iota : \mathcal{L} \rightarrow U(\mathcal{L})$ from \mathcal{L} to its enveloping (vertex) algebra and a unique (vertex) algebra homomorphism $\psi : U(\mathcal{L}) \rightarrow A$, such that $\psi \circ \iota = \phi$ (cf. [Hum]).

The classical Lie algebra’s enveloping algebra has the structure not only of a bialgebra, but a Hopf algebra. In the vertex Lie algebra setting, a general vertex bialgebra or vertex Hopf algebra structure has yet to be defined. However, we will show that given a graded vertex Lie algebra, Primc’s enveloping vertex algebra is naturally a grade vertex coalgebra. This is a step toward giving the enveloping vertex algebra a general vertex bialgebra structure. (In [L], Li defines a structure with a vertex algebra and a coassociative coalgebra structure, but points out there are more general constructions.)

We begin by rigorously defining vertex Lie algebra, adding a grading and constructing the (now graded) enveloping vertex algebra. Our definition will make use of the symbol \simeq which indicates equality of the principal parts of two formal Laurent series. For example, given two Laurent series $f(x_1, x_2)$ and $g(x_1, x_2)$ in formal variables x_1 and x_2 , $f \simeq g$ means that for all $j, k \in \mathbb{Z}_+$ the coefficient of $x_1^{-j}x_2^{-k}$ in f is equal to the same coefficient in g . Note below that in addition to adding grading, we are restricting Primc’s work over vertex superalgebras to the vertex algebra setting.

Definition 8.1. *A vertex Lie algebra is a vector space V equipped with a linear operator $D : V \rightarrow V$ (called the derivation) and a linear map*

$$Y(\cdot, x) : V \rightarrow (End V)[[x^{-1}]]$$

$$v \mapsto Y(v, x) = \sum_{k \in \mathbb{N}} v_k x^{-k-1} \quad (\text{where } v_n \in End V),$$

satisfying the following axioms for $u, v \in V$:

1. *Truncation:* $u_k v = 0$ for k sufficiently large.
2. *Half Jacobi identity:*

$$(8.1) \quad x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1)$$

$$\simeq x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2).$$

3. *Half skew-symmetry:* $Y(u, x)v \simeq e^{xD}Y(v, -x)u$.
4. *D-bracket:* $[D, Y(u, x)] = Y(Du, x) = \frac{d}{dx}Y(u, x)$.

Certainly any vertex algebra is a vertex Lie algebra with $D(v) = v_{-2}\mathbf{1}$. A vertex Lie algebra may also carry a natural \mathbb{Z} -grading.

Definition 8.2. Let $V = \coprod_{k \in \mathbb{N}} V_{(k)}$ be a \mathbb{Z} -graded vector space.. A vertex Lie algebra structure on V will be said to be graded if for each $r, s, t \in \mathbb{N}$, $u \in V_{(r)}$, and $v \in V_{(s)}$,

$$u_t v \in V_{(r+s-t-1)},$$

and

$$Du \in V_{(r+1)}.$$

This definition agrees with the notion of grading for vertex (operator) algebras. Now consider the affinization of a vertex Lie algebra V with derivation D , written $V \otimes \mathbb{F}[q, q^{-1}]$. For graded vertex Lie algebras, $V \otimes \mathbb{F}[q, q^{-1}]$ carries a natural \mathbb{Z} -grading by letting q^k have weight $-k - 1$. Notationally, given $v \in V$ and $n \in \mathbb{Z}$, we use the abbreviation

$$v_n = v \otimes q^n.$$

Primc shows that the quotient space

$$\mathcal{L}(V) = (V \otimes \mathbb{F}[q, q^{-1}]) / \{(Dv)_n + n v_{n-1} | v \in V, n \in \mathbb{Z}\}$$

is a Lie algebra under the bracket

$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i},$$

for $u, v \in V$ and $m, n \in \mathbb{Z}$, where the images of $u_m, v_n \in V \otimes \mathbb{F}[q, q^{-1}]$ in $\mathcal{L}(V)$ are again denoted u_m and v_n , respectively. (Employing the notation of [P], $(u_i v)_{m+n-i}$ means the image of $u_i v \otimes q^{m+n-i}$.)

Since $(Dv)_n$ has the same weight as v_{n-1} for any homogeneous $v \in V$, $\mathcal{L}(V)$ is \mathbb{Z} -graded. Further, $D : \mathcal{L}(V) \rightarrow \mathcal{L}(V)$ given by $D(v_n) = (Dv)_n$ is a grading preserving derivation of the Lie algebra. Finally, given $k, \ell, m, n \in \mathbb{Z}$, $u \in V_{(k)}$ and $v \in V_{(\ell)}$, $(u_i v)_{m+n-i}$ has weight $k + \ell - m - n - 2$ for any i , which is the weight that $[u_m, v_n]$ must have in order to make the quotient $\mathcal{L}(V)$ a \mathbb{Z} -graded Lie algebra with a grading preserving derivation.

Moving forward, $\mathcal{L}(V)$ splits into two D -invariant Lie subalgebras,

$$\begin{aligned} \mathcal{L}_-(V) &= \text{span}\{v_n | v \in V, n < 0\}, \\ \mathcal{L}_+(V) &= \text{span}\{v_n | v \in V, n \geq 0\}. \end{aligned}$$

The enveloping vertex algebra we seek is a generalized Verma module induced from a trivial $\mathcal{L}_+(V)$ -module \mathbb{F} :

$$\mathcal{V}(V) = U(\mathcal{L}(V)) \otimes_{U(\mathcal{L}_+(V))} \mathbb{F},$$

where U gives the enveloping algebra of a given Lie algebra. Primc established that $\mathcal{V}(V)$ is a vertex algebra, with the derivation D on $\mathcal{L}(V)$ extending naturally to $\mathcal{V}(V)$ and with vacuum vector, $\mathbf{1}$, the tensor of $1 \in U(\mathcal{L}(V))$ and $1 \in \mathbb{F}$.

Again, if V is \mathbb{Z} -graded then the grading is preserved in $\mathcal{V}(V)$ and we write

$$\mathcal{V}(V) = \prod_{k \in \mathbb{Z}} \mathcal{V}(V)_{(k)}.$$

Exploiting the natural vector space isomorphism

$$\mathcal{V}(V) \cong U(\mathcal{L}_-(V)),$$

the elements of $\mathcal{V}(V)$ may be written as sums of homogeneous elements of the form

$$\hat{v} = v_{n_1}^{(1)} v_{n_2}^{(2)} \cdots v_{n_t}^{(t)} \in \mathcal{V}(V)_{(p_1 + \cdots + p_t - n_1 - \cdots - n_t)}$$

where $t \in \mathbb{N}$; $n_1, n_2, \dots, n_t \in \mathbb{Z}_-$; $v^{(1)} \in V_{(p_1)}, v^{(2)} \in V_{(p_2)}, \dots, v^{(t)} \in V_{(p_t)}$ for some $p_1, p_2, \dots, p_t \in \mathbb{N}$; and tensor products are omitted from the notation. The derivation and the product operation both naturally preserve the grading, as they are both induced from $\mathcal{L}(V)$. (This motivates the above choice of grading: $\text{wt } v_n = \text{wt } v - n - 1$.) Thus $\mathcal{V}(V)$ is a graded vertex algebra and by Corollary 7.2, $\mathcal{V}(V)$ also carries the structure of a graded vertex coalgebra.

The map $\iota_V : V \rightarrow \mathcal{L}_-(V)$ given by $\iota_V(v) = v_{-1}$ is a vector space isomorphism by Theorem 4.6 in [P]. Thus, if V is finite dimensionally graded, as is the case for vertex operator algebras, $\mathcal{L}_-(V)$ is finite dimensionally graded and as a consequence $\mathcal{V}(V)$ is a finite dimensionally graded vertex algebra and coalgebra. By the discussion at the beginning of Section 8, it is reasonable to conclude that this is the ‘appropriate’ multiplicative structure for an enveloping vertex bialgebra, and by the discussion in Section 7, it is reasonable to conclude that this is the appropriate comultiplicative structure.

REFERENCES

- [B] R. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proceedings of the National Academy of Science USA **83** (1986), 3068-3071.
- [BPZ] A. Belavin, A. Polyakov, A. Zamolodchikov, *Infinite conformal symmetries in two-dimensional quantum field theory*, Nuclear Physics, **B241** (1984), 333-380.
- [Bo] Lev Borisov, *Chiral rings of vertex algebras of mirror symmetry*, Math. Z. **248** (2004), no. 3, 567-591.
- [DJM] Esturō Date, Michio Jimbo, Tetsuji Miwa, *Solitons: Differential equations, symmetries and infinite-dimensional algebras*, trans. Miles Reid, Cambridge Tracts in Mathematics, 135, Cambridge University Press, Cambridge, 2000.
- [FF1] Boris Feigin, Edward Frenkel, *Integrals of motion and quantum groups*, Lect. Notes in Math. **1620** (1996), 349-418.
- [FF2] Boris Feigin, Edward Frenkel, *Kac-Moody groups and integrability of soliton equations*, Invent. Math. **120** (1995), no. 2, 379-408.
- [FF3] Boris Feigin, Edward Frenkel, *Integrable hierarchies and Wakimoto modules*. Differential topology, infinite-dimensional Lie algebras, and applications, Amer. Math. Soc. Transl. **194** (1999), Series 2, 27-60.
- [F] Edward Frenkel, *Vertex algebras and algebraic curves*, Seminaire Bourbaki, 52nd year (1999-2000), num. 875.
- [FB] Edward Frenkel, David Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs, 88, American Mathematical Society, Providence, Rhode Island, 2001.
- [FHL] Igor B. Frenkel, Yi-Zhi Huang, James Lepowsky, *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, Memoirs of the AMS, Number 494, Providence, Rhode Island, July 1993.
- [FK] Igor Frenkel, Victor Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), 23-66.
- [FLM] Igor Frenkel, James Lepowsky, Arne Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, Inc., San Diego, 1988.
- [Fri] Daniel Friedan, *Nonlinear models in $2+\epsilon$ dimensions*, Ann. Phys. **163** (1985), no. 2, 318-419.
- [GM] Vassily Gorbounov, Fyodor Malikov, *Vertex algebras and the Landau-Ginzburg/Calabi-Yau correspondence*, Mosc. Math. J. **4** (2004), no. 3, 729-779, 784.
- [GMS] Vassily Gorbounov, Fyodor Malikov, Vadim Schechtman, *Gerbes of chiral differential operators. II. Vertex algebroids*, Invent. Math. **155** (2004), no. 3, 605-680.
- [GSW] Michael Green, John Schwarz, Edward Witten, *Superstring theory*, Vol. 1. Introduction, Second edition, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1998.
- [H] Yi-Zhi Huang, *Two-Dimensional Conformal Geometry and Vertex Operator Algebras*, Birkhäuser, Boston, 1997.
- [H2] Yi-Zhi Huang, *Differential equations, duality, and modular invariance*, preprint, arXiv:math.QA/0303049.

- [Hub1] Keith Hubbard, The notion of a vertex operator coalgebra: a construction and geometric characterization, Ph.D. thesis, University of Notre Dame, 2005, <http://etd.nd.edu/ETD-db/theses/available/etd-03282005-121128/unrestricted/HubbardK032005.pdf>.
- [Hub2] Keith Hubbard, The notion of vertex operator coalgebra and a geometric interpretation, *Commun. Algebra* **34** (2006), 1541 - 1589.
- [Hub3] Keith Hubbard, *Constructions of vertex operator coalgebras via vertex operator algebras*, *Journal of Algebra* **294** (2005), 278-293.
- [Hub4] Keith Hubbard, *The duality between vertex operator algebras and coalgebras, module and comodules*, *Contemporary Mathematics* **442** (2007), 339-354.
- [Hum] James E. Humphreys, *Introduction to Lie Algebras and Representations Theory*, Springer, New York, 1972.
- [J] Vaughan F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, *Ann. Math.* **126** (1987), 335-388.
- [K] Victor Kac, *Vertex algebras for beginners*, University Lecture Series **10**, Amer. Math. Soc., 1998.
- [L] Hai-sheng Li, *A smash product construction of nonlocal vertex algebras*, *Communications in Contemporary Mathematics* **9** (2007), 605-637.
- [LL] James Lepowsky, Haisheng Li, *Introduction to Vertex Operator Algebras and Their Representations*, Birkhäuser, Boston, 2004.
- [LW] James Lepowsky, Robert L. Wilson, *Construction of the affine Lie algebra $A_1^{(1)}$* , *Comm. Math. Phys.* **62** (1978), 43-53.
- [M] Susan Montgomery, *Hopf Algebras and Their Actions on Rings*, Regional Conference Series in Mathematics, American Mathematical Society, Providence, Rhode Island, 1993.
- [P] Mirko Primc, *Vertex algebras generated by Lie algebras*, *J. Pure Appl. Algebra*, vol. 135, no. 3, pp. 253-293, 1999.
- [RT] Nicolai Reshetikhin, Vladimir Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, *Invent. Math.* **103** (1991), 547-597.
- [S] Graeme Segal, *Unitary representations of some infinite-dimensional groups*, *Commun. Math. Phys.* **80** (1981), 301-342.
- [ST] Stephan Stolz, Peter Teichner, *What is an elliptic object?*, Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal, U. Tillmann (ed.), Cambridge University Press, 2004.
- [U] Vladimir Ufnarovskij, *Combinatorial and asymptotic methods in algebra*, Encyclopaedia of Mathematical Sciences 57, Algebra VI, AI Kostrikin and IR Shafarevich, Eds., Springer-Verlag, Berlin, 1995.
- [W] Edward Witten, *Quantum field theory and the Jones polynomial*, *Comm. Math. Phys.* **121** (1989), 351-399.
- [Z] Yongchang Zhu, *Modular invariance of characters of vertex operator algebras*, *J. Amer. Math. Soc.* **9** (1996), 237-307.