

# Research Statement

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## Part 1. Introduction

I expect to receive my Ph.D. from the University of Notre Dame in May 2005 for my research in vertex operator algebra theory. My dissertation examines the coalgebraic structure arising from the geometry of genus-zero worldsheets in conformal field theory. I would like to continue this work and am particularly interested in extending the known structure of vertex operator algebras and their representations to bialgebraic and Hopf algebraic structures as a continuation of my work in extending them to the coalgebraic structure. I would also like to understand to what extent the conformal geometry underlying these structures is independent of the multiplicative and comultiplicative structures themselves.

Vertex algebras and vertex operator algebras (VOAs) are one of the exciting new fields in modern algebra and, although much progress has been made since their inception, their intricate structure has made vertex algebras and VOAs two of the more challenging mathematical objects to classify and generalize. The definitions of vertex algebra and of VOA were partly motivated by the structures one would expect from genus-zero conformal field theory.

Conformal field theory, or more specifically string theory, seeks to unify all the fundamental forces in nature into a single theory by modeling particles as one-dimensional objects (strings) moving through space-time, and then encoding this model in an algebraic context [FS, S2, V]. It is an attempt to resolve the decades long impasse between quantum mechanics and relativity. Vertex algebras were first introduced in [B] and describe constraints on the multiplicative structure induced by modeling two strings combining into one as they travel through space-time. The notion of a VOA was then introduced in [FLM], which interpreted vertex algebras in a formal calculus setting with a generating function greatly generalizing the classical Jacobi identity and, perhaps most significantly, added a compatible conformal structure.

VOAs are connected to understanding the Monster group (the largest sporadic finite simple group) [FLM], representation theory and infinite-dimensional Lie algebras [LW, FK, S1], modular functions and modular forms [Z, H3], Calabi-Yau manifolds [Fri, GSW], infinite-dimensional integrable systems [LW, DJM], knot and three-manifold invariants [J, RT, W] and elliptic cohomology [ST]. Among other accolades the field has received, Borcherds received the Fields medal in 2000 for his work in monstrous moonshine, a collection of mysterious connections between the theory of VOAs and modular functions in number theory. A deeper and broader understanding of VOAs would not only be a boon for the larger mathematics community, but also for much of the theoretical physics community.

Although VOAs had physical origins in the geometry of string theory, a rigorous correspondence between the algebraic formalism and the geometric motivation was not completed until Huang's work [H1]. The basic data of a VOA includes an infinite family of multiplication operators and a representation of the Virasoro algebra on an infinite dimensional vector space, along with a number of intricate formal calculus axioms. Motivated by [F], Huang described the multiplication and interpreted the representation of the Virasoro algebra from the geometric path swept out as  $n$  strings combine in space-time [H1].

As a student of Katrina Barron's at Notre Dame, I have developed the algebraic theory arising from the path swept out by splitting strings in space-time [Hub1-Hub3]. My dissertation [Hub1] examines the geometric motivation arising from the model of one string splitting in space-time and the induced comultiplicative structure, motivating a structure I called a vertex operator coalgebra (VOC). I have also constructed a class of examples of VOCs [Hub1, Hub3], discussed an algebraic correspondence between certain VOCs and VOAs [Hub3], defined a more general notion called a vertex coalgebra [Hub4], and described the derivation, commutativity, associativity, skew-symmetry and modules properties of both vertex coalgebras and VOCs [Hub4].

My past research opens doors for further investigation in a number of directions, but I will elaborate on three immediate questions. First, from a purely algebraic perspective, having an algebraic structure (or in our case a multiplicative structure that satisfies a kind of generalization of the classical Jacobi identity) and a coalgebraic structure (similarly a comultiplicative structure satisfying a different Jacobi type identity) cries out to have the bialgebraic theory developed (particularly in the vertex tensor categories setting). Second, having a multiplication and a comultiplication allows for the consideration of antipodal maps (in the Hopf algebraic sense) and investigation of the corresponding module theory. This question can actually be considered independent of the first but answers to both obviously fit together well. Third, it would be useful to establish how much vertex tensor categories and the corresponding module tensors of VOAs depend on the conformal geometry and to what degree these are independent structures. This statement will outline my past research and how I intend to engage these new problems.

## **Part 2. Past Research and Methods Employed**

My dissertation [Hub1] examines the geometric motivation and comultiplicative structure arising from one string splitting in space-time and the induced comultiplicative structure. This comultiplication, along with its interaction with a representation of the Virasoro algebra gave rise to a structure called a vertex operator coalgebra (VOC). A VOC, which I describe in the context of formal calculus [Hub1, Hub2], bears a strong resemblance to a VOA (in the sense of duality) but with changes to the truncations, the representation of the Virasoro algebra and the commutativity properties that would have been difficult to discover without delving into the conformal geometry. In order to reach the formal calculus and representation theoretic setting, it was necessary to work with representations of the Virasoro Lie algebra both on geometric spaces and on arbitrary infinite-dimensional vector spaces. The argument began by considering worldsheets, or two-dimensional genus-zero Riemann manifolds with half-infinite tubes attached, representing the paths swept out by strings moving through space-time. The worldsheets were then explored and characterized according to conformal equivalence classes, using infinitesimal local coordinates. Interpreting the local coordinate maps in terms of generators of the geometric tangent space at each puncture allowed for the choice and convenient description of canonical representatives as well as a description of composition, or sewing, of conformal equivalence classes of worldsheets. After describing equivalence classes of worldsheets, my thesis described a natural partial operadic structure on these equivalence classes and then described a kind of coalgebraic structure over that operad. Specifically, geometric vertex operator algebras (GVOCs) may be characterized as meromorphic morphisms (or as coalgebras) of a  $\mathbb{C}$ -extension of the partial operad of conformal equivalence classes of spheres with outgoing tubes to the partial pseudo-operad  $\{\mathcal{H}_V(1, n)\}_{n \in \mathbb{N}}$ , that consists of appropriately defined  $n$ -dimensional comultiplications. Characterizing these objects in a purely algebraic context was the main purpose of my thesis. The notion of vertex operator coalgebra is meant to interpret the operadic and conformal geometric notion of GVOC in a context of formal calculus and formal variables, i.e. putting them in a context where convergence and meromorphicity is no longer at issue. The main proof of my thesis shows that the category of VOCs is not only equivalent but isomorphic to the category of GVOCs, thus rigorizing the correspondence between worldsheet geometry of splitting strings and the algebraic structures formally describing the correlation functions of such interactions.

As a necessary step in proving the above mentioned isomorphism, a good deal of VOC theory had to be developed, including numerous derivation properties and an equivalent characterization of the generalized Jacobi identity. However, the VOC structure is intricate enough that even this additional description did not immediately illuminate what examples of VOCs should look like. The single greatest step toward that result is in [Hub3] (and also included in [Hub1]) where I was first able to constructively show the existence of VOCs. This paper constructs VOCs via VOAs equipped with nondegenerate Virasoro preserving bilinear forms and in the process does much to explain one way in which VOCs may be thought of as dual to VOAs. Also, in [Hub3] I explicitly construct VOCs corresponding to Heisenberg algebras (that is, corresponding to

free bosons in conformal field theory) and I characterize the unique bilinear forms (up to scalar) that exist in that context.

Since the concept of vertex algebras [B] preceded that of VOAs, a natural question to ask was whether the conformal structure could be omitted from VOCs and the concept of vertex coalgebra introduced. This is the topic of my third paper [Hub4], which also develops the module theory of vertex coalgebras and VOCs. Additionally, it discusses the concepts of derivations, (weak) associativity, (weak) commutativity, and skew-symmetry in the coalgebraic context.

### Part 3. Research Objectives and Methods

Three areas of future study seem particularly relevant and achievable considering my past experience. Solutions to these problems would give better insight into the extent to which vertex algebras and VOAs mirror the behavior of Lie algebras, would extend our understanding of how the theory of tensor products of VOAs behaves in a vertex tensor categorical context [HL1-HL4, H2] and could extend our mathematical understanding of the modules involved in two-dimensional conformal field theory.

#### Developing vertex bialgebraic and vertex operator bialgebraic theory

In [Hub3], a vertex operator algebra with an appropriate bilinear form can be given the structure of a vertex operator coalgebra. The converse is also true and I am currently preparing these results for publication. However, as motivated by the theory of Lie algebras, one of the most fundamental questions to ask is how multiplicative and comultiplicative structures may be combined to achieve a bialgebraic structure. Achieving a vertex (operator) bialgebra necessitates verifying that the comultiplication, (co-Virasoro,) and counit maps are vertex (operator) algebra morphisms. This task is substantially more daunting than it might first appear. Let  $Y(-, x)- : V \otimes V \rightarrow V[[x, x^{-1}]]$  be a standard vertex algebra multiplication (cf. [FLM]) so that

$$Y(u, x)v = \sum_{k \in \mathbb{Z}} u_k v x^{-k-1} \in V((x)).$$

Also let  $\mathcal{A}(x)- : V \rightarrow (V \otimes V)[[x, x^{-1}]]$  ( $\mathcal{A}$  is called the *coproduct*) be a standard vertex coalgebra multiplication (cf. [Hub4]) so that

$$\mathcal{A}(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v) x^{-k-1} \in (V \otimes V)((x^{-1})).$$

Composition of these operators is not in general well defined unless different formal variables are used, i.e.  $\mathcal{A}(x_2)Y(u, x_1)v$ . But then depending on the order of compositions of  $Y$  and  $\mathcal{A}$  the result will be in different fields. Thus the use of vertex tensor categories. Vertex tensor categories as discussed in [HL1-HL4, H2] provide a framework in which to construct a correctly modified comultiplication so that it will be compatible with the existing multiplicative structure. These categories, by necessity depend heavily on the conformal geometry of spheres with tubes and compositions thereof. Standard braided tensor categories neglect conformal structure and are a special case of vertex tensor categories. Constructing bialgebraic objects in the vertex tensor categorical setting will also be useful in directly constructing the tensor products themselves. Again we use Lie algebras as a guide. If  $M$  and  $N$  are modules of a Lie algebra  $L$ , then the tensor product  $M \otimes N$  is naturally a module by considering that

$$\begin{aligned} a \cdot (m \otimes n) &= \Delta(a) \cdot (m \otimes n) \\ &= (a \otimes 1 + 1 \otimes a) \cdot (m \otimes n) \\ &= (a \cdot m \otimes n) + (m \otimes a \cdot n) \end{aligned}$$

where  $a \in L, m \in M, n \in N$  and  $\Delta$  is comultiplication in the universal enveloping algebra  $U(L)$ . Further, associativity of tensor product modules follows from the coassociativity of comultiplication in  $U(L)$ . Conceptually, the same type of argument should work in vertex tensor categories to replace the current tensor description which is described by taking an action on a subspace of the dualized algebraic tensor, then taking the contragredient module. A non-rigorous attempt at such a comultiplication was given in [MS], but was based on an underlying vector space that does not in general have the size or weight characteristics one would want in such a structure.

Understanding the module theory of vertex algebras and VOAs has lead to a greater understanding of connections with mathematics and physics at large. Huang and Lepowsky have succeeded in interpreting modules over certain VOAs in a vertex tensor category context, to some degree replacing specialized pieces with a general theory. It is hoped that this general theory can answer such long standing questions as how the moonshine module associated to the Monster group can have the simplest possible monodromy of correlation functions (trivial) and yet the richest possible finite symmetries [D, G, HL4]. It is likely that in addition to modules of vertex algebras and VOAs, modules of vertex coalgebras and VOCs fit into a larger representation theoretic picture once in the context of vertex tensor categories.

### Studying antipodal maps

Another compelling question is whether there is a natural antipodal map between appropriately defined multiplication and comultiplication, i.e. is there an antipodal map  $S : V \rightarrow V$  such that the convolution map of  $S$  with the identity will yield the composition of the counit and unit?

Again, because of formal variables and the fact that the terms multiplication and comultiplication actually refer to infinite families of multiplications and comultiplications, this question is more challenging than the classical case. Of course, combined with a successful solution to my above mentioned research project on bialgebraic structure, this would give a Hopf algebraic structure on vertex bialgebras and vertex operator bialgebras. Luckily special cases of vertex (operator) algebras seem to circumvent the question of a bialgebra structure and indicate a fruitful method of pursuing antipodal maps. These cases indicate that given the multiplication operator  $m_{-1}$  of degree 0 from a vertex (operator) algebra (i.e. the coefficient of  $x^0$  in  $Y(u, x)v$ , which is just a multiplication) and the comultiplication operator  $\Delta_{-1}$  of degree 0 from a vertex (operator) coalgebra (i.e. the coefficient of  $x_0$  in  $\mathcal{A}(x)v$ ), we would hope that the antipodal map would satisfy

$$m_{-1} \circ (S \otimes Id_V) \Delta_{-1}(v) = \mathbf{1}c(v) = m_{-1} \circ (Id_V \otimes S) \Delta_{-1}(v).$$

For Heisenberg vertex operator algebras and coalgebras, which correspond to bosons freely interacting in the physical universe, this equation indirectly yields a unique inductive definition of  $S$ . Even in the one-dimensional Heisenberg case, this definition becomes so combinatorially intricate that even employing computational software only reveals small pieces of the information. Using Maple for computations, I have used computations to predict relations that should occur in general. These are relations which I hope to explore as a part of my research program.

In addition to understanding particular examples, I would also like to investigate when antipodes exists and how they may be defined in general. The antipodal map is related to the construction of contragredient modules, which appear in [HL1] and should give direction in how to pursue the overarching antipodal map questions. A more general approach seems likely to be found in the context of vertex tensor categories and it is possible that closed form combinatorial descriptions will appear in the general setting as well.

### Understanding the independence of conformal geometric structure in vertex tensor categories

It is significant to note that while the notion of VOA generalizes directly to that of vertex algebra, vertex tensor categories as now understood have no generalization to vertex algebras because the tensor

description is heavily dependent upon the conformal structure. I would like to investigate the question of how independent the conformal structure in vertex tensor categories is of the multiplicative, comultiplicative and module tensor structure.

As mentioned before, in VOAs and generally in vertex tensor categories the current understanding of tensors of modules depends on using the contragredient module. Given a module  $W$  of a VOA  $V$ , the graded dual of  $W$  is denoted  $W'$ . Contragredient vertex operators are a family of operators on the graded dual,  $Y'(-, x) : V \rightarrow \text{End}(W')[[x, x^{-1}]]$ , determined by the condition

$$\langle Y'(v, x)w', w \rangle = \langle w', Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \rangle$$

where  $v \in V$ ,  $w \in W$ ,  $w' \in W'$ , and  $L(1)$  and  $L(0)$  are the weight  $-1$  and weight  $0$  operators, respectively, of the representation of the Virasoro algebra on  $V$ . The conformal subtlety is wrapped up in the fact that this definition is totally dependent on the definition of the representation of the Virasoro algebra, which encodes the conformal structure. In the context of vertex algebras, the conformal structure can be confidently omitted without any inherent change to the skew-symmetry, commutativity and associativity properties of multiplications. But the above description does not allow for that separation. A clearly understanding of why the conformal structure is involved here and whether that structure is necessary or just one avenue to tensors would be very useful.

Thus far, I have been able to interpret the notion of contragredient operator in the coalgebraic setting, also using conformal structure. However, the two have yet to be combined or interpreted in any coherent unified picture. I find this an exciting research prospect.

Finally, the projects outlined here should serve as a springboard for extended research into a number of different fields (depending on what those results are) from modular functions and elliptic cohomology to number theoretic questions.

#### Part 4. Conclusion

I have greatly enjoyed the process of entering the mathematical research community and the thoroughfare of ideas that is modern mathematics. I look forward to becoming a contributing member of a new research community – listening to colleagues' progress and sharing my own, and am hopeful that that community could extend beyond just the pure mathematics community. The particular strengths of the community where I get to work will significantly shape my longer term research program. I am excited about the prospect of engaging a new research community and look forward to collaborating with fellow mathematicians.

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