CONSTRUCTIONS OF VERTEX OPERATOR COALGEBRAS VIA VERTEX OPERATOR ALGEBRAS

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Abstract. The notion of vertex operator coalgebra is presented which corresponds to the family of correlation functions modeling one string propagating in space-time splitting into n strings in conformal field theory. This notion is in some sense dual to the notion of vertex operator algebra. We prove that any vertex operator algebra equipped with a non-degenerate, Virasoro preserving, bilinear form gives rise to a corresponding vertex operator coalgebra. We then explicitly calculate the vertex operator coalgebra structure and unique bilinear form for the Heisenberg algebra case, which corresponds to considering free bosons in conformal field theory.

1. Introduction

The theory of vertex operator algebras has been an ever expanding field since its inception in the 1980s when Borcherds first introduced the precise notion of a vertex algebra [B]. The notion of a vertex operator algebra (VOA) was then introduced in [FLM], which interpreted vertex algebras in a formal calculus setting with a generating function greatly generalizing the classical Jacobi identity and, perhaps most significantly, added a compatible conformal structure. VOAs are connected to understanding the Monster group (the largest sporadic finite simple group) [FLM], representation theory and infinite-dimensional Lie algebras [LW, FK, S1], modular functions and modular forms [Z, H3], Calabi-Yau manifolds [Fri, GSW], infinite-dimensional integrable systems [LW, DJM], knot and three-manifold invariants [J, RT, W] and elliptic cohomology [ST].

VOAs have been interpreted as a specific type of algebra over an operad [HL1] and, quite recently, in [Hub1] and [Hub2], the algebraic structure induced by considering the algebraic structure (in the operad sense) of worldsheets swept out by closed strings propagating through space-time, has been supplemented by examining the induced coalgebraic structure which gives rise to vertex operator coalgebras (VOCs). The notion of VOC corresponds to the coalgebra of correlation functions modeling one string splitting into n strings in space-time, whereas VOAs correspond to the algebra of correlation functions of n strings combining into one string in space-time. Not only are the notions of VOC and VOA integral parts of conformal field theory but the successful integration of the two notions might provide insight into the representation theory of vertex operator algebras, and in particular the structure of vertex tensor categories [HL2].

Examples of VOAs have been quite useful, both in understanding VOA structure and also in understanding other objects (particularly the Monster finite simple group [FLM]), but have historically been quite challenging to construct. Building a

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substantial pool of examples has taken decades. (See [LL] for one recent list of constructions.) One might expect similar challenges in generating examples of VOCs. However, in this paper, via an appropriately defined bilinear form, we generate a large family of examples of VOCs by tapping into the extensive work on VOA examples. We will also explicitly calculate the family of examples corresponding to Heisenberg algebras and demonstrate the naturally adjoint nature of Heisenberg VOAs and VOCs. The Heisenberg example corresponds to considering free bosons in conformal field theory.

Finally, dating back to the 1980s conformal field theories have included axioms about the adjointness of operators induced by manifolds of opposite orientation (cf. [S2]). The construction in this paper indicates that VOC operators satisfy this condition and are adjoint to VOA operators when an appropriate bilinear form exists.

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2. Definitions and algebraic preliminaries

We begin by reviewing a necessary series from the calculus of formal variables, then recall the definition of vertex operator coalgebra (cf. [Hub1], [Hub2]) and vertex operator algebra (cf. [FLM], [FHL]).

2.1. **Delta functions.** We define the "formal δ-function" to be

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n.$$

Given commuting formal variables x_1, x_2 and n an integer, $(x_1 \pm x_2)^n$ will be understood to be expanded in nonnegative integral powers of x_2. (This is the convention throughout vertex operator algebra and coalgebra theory.) Note that the δ-function applied to $\frac{x_1-x_2}{x_0}$, where x_0, x_1 and x_2 are commuting formal variables, is a formal power series in nonnegative integral powers of x_2 (cf. [FLM], [FHL]).

2.2. **The notion of vertex operator coalgebra.** The following description of a vertex operator coalgebra is the central structure of this paper. Originally motivated by the geometry of propagating strings in conformal field theory ([Hub1], [Hub2]), VOCs may be formulated in terms of vectors spaces over an arbitrary characteristic zero field $\mathbb{F}$ using formal commuting variables x, x_1, x_2, x_3.

Definition 2.1. *A* vertex operator coalgebra (over $\mathbb{F}$) of rank $d \in \mathbb{F}$ *is a $\mathbb{Z}$-graded vector space over* $\mathbb{F}$*,*

$$V = \coprod_{k\in\mathbb{Z}} V_{(k)},$$

such that $\dim V_{(k)} < \infty$ *for* $k \in \mathbb{Z}$ *and* $V_{(k)} = 0$ *for k sufficiently small, together with linear maps*

$$\begin{aligned} \Lambda(x) : V &\to (V \otimes V)[[x, x^{-1}]] \\ v &\mapsto \Lambda(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v) x^{-k-1} \qquad (\textit{where } \Delta_k(v) \in V \otimes V), \end{aligned}$$

$$c : V \to \mathbb{F},$$

$$\rho : V \to \mathbb{F},$$

called the coproduct*, the* covacuum map *and the* co-Virasoro map*, respectively, satisfying the following 7 axioms:*

1. *Left counit: For all $v \in V$*

$$(c \otimes Id_V)\Lambda(x)v = v \tag{2.1}$$

2. *Cocreation: For all $v \in V$*

$$(Id_V \otimes c)\Lambda(x)v \in V[[x]] \quad \textit{and} \tag{2.2}$$

$$\lim_{x \to 0} (Id_V \otimes c)\Lambda(x)v = v. \tag{2.3}$$

3. *Truncation: Given $v \in V$, then $\Delta_k(v) = 0$ for k sufficiently small.*
4. *Jacobi identity:*

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)(Id_V \otimes \Lambda(x_2))\Lambda(x_1) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)(T \otimes Id_V) \\ (Id_V \otimes \Lambda(x_1))\Lambda(x_2) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)(\Lambda(x_0) \otimes Id_V)\Lambda(x_2). \end{aligned} \tag{2.4}$$

5. *Virasoro algebra: The Virasoro algebra bracket,*

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k} d,$$

holds for $j, k \in \mathbb{Z}$, where

$$(\rho \otimes Id_V)\Lambda(x) = \sum_{k \in \mathbb{Z}} L(k) x^{k-2}. \tag{2.5}$$

6. *Grading: For each $k \in \mathbb{Z}$ and $v \in V_{(k)}$*

$$L(0)v = kv. \tag{2.6}$$

7. *$L(1)$-derivative:*

$$\frac{d}{dx}\Lambda(x) = (L(1) \otimes Id_V)\Lambda(x). \tag{2.7}$$

We denote this vertex operator coalgebra by $(V, \curlyvee, c, \rho)$ or simply by V when the structure is clear.

Note that $\curlyvee$ is linear so that, for example, $(Id_V \otimes \curlyvee(x_1))$ acting on the coefficients of $\curlyvee(x_2)v \in (V \otimes V)[[x_2, x_2^{-1}]]$ is well defined. Notice also, that when each expression is applied to any element of V, the coefficient of each monomial in the formal variables is a finite sum.

2.3. **The definition of a VOA.** Vertex operator algebras were first defined in [FLM], but it was not until [H1], [H2] that this definition was rigorously tied to the geometry of conformal field theory. This correspondence, along with its operadic interpretation in [HL1], helped to motivate the notion of VOC in [Hub1]. It is not surprising then, that the axioms of VOA and VOC would strongly resemble each other.

Definition 2.2. *A* vertex operator algebra (over $\mathbb{F}$) of rank $d \in \mathbb{F}$ *is a $\mathbb{Z}$-graded vector space over $\mathbb{F}$,*

$$V = \coprod_{k \in \mathbb{Z}} V_{(k)},$$

such that $\dim V_{(k)} < \infty$ *for $k \in \mathbb{Z}$ and $V_{(k)} = 0$ for k sufficiently small, together with a linear map $V \otimes V \to V[[x, x^{-1}]]$, or equivalently,*

$$\begin{aligned} Y(\cdot, x) : V &\to (End\ V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{k \in \mathbb{Z}} v_k x^{-k-1} \qquad (\text{where } v_n \in End\ V), \end{aligned}$$

and equipped with two distinguished homogeneous vectors in V, $\mathbf{1}$ (the vacuum*) and ω (the* Virasoro element*), satisfying the following 7 axioms:*

1. Left unit: For all $v \in V$

$$Y(\mathbf{1}, x)v = v \tag{2.8}$$

2. Creation: For all $v \in V$

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \tag{2.9}$$

$$\lim_{x \to 0} Y(v, x)\mathbf{1} = v. \tag{2.10}$$

3. Truncation: Given $v, w \in V$, then $v_k w = 0$ for k sufficiently large.

4. Jacobi Identity: For all $u, v \in V$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2). \end{aligned} \tag{2.11}$$

5. Virasoro Algebra: The Virasoro algebra bracket,

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}d,$$

holds for $j, k \in \mathbb{Z}$, where

$$Y(\omega, x) = \sum_{k\in\mathbb{Z}} L(k)x^{-k-2}. \tag{2.12}$$

6. Grading: For each $k \in \mathbb{Z}$ and $v \in V_{(k)}$

$$L(0)v = kv. \tag{2.13}$$

7. $L(-1)$-Derivative: Given $v \in V$,

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x). \tag{2.14}$$

We denote a VOA either by V or by the quadruple $(V, Y, \mathbf{1}, \omega)$. A vector $v \in V_{(k)}$ for some $k \in \mathbb{Z}$ is said to be a *homogeneous vector of weight* k and we write wt $v = k$. A pair of basic properties of VOAs will be necessary for our discussion (cf. [FHL], [LL]):

$$\begin{aligned} Y(v, x)\mathbf{1} &= e^{xL(-1)}v & &\text{for } v \in V, \\ \text{wt } v_k w &= r + s - k - 1 & &\text{for } v \in V_{(r)},\ w \in V_{(s)} \end{aligned}$$

Remark 2.3. *Note that while the axioms of VOC and VOA strongly resemble each other, there are several important differences. First, whereas the creation and truncation axioms of VOAs both bound the power of the formal variable from below, in VOCs cocreation allows only non-negative powers of x while truncation allows only finitely many positive powers of x but infinitely many negative powers. Among other implications, this means that cocreation actually generates polynomials in x. Second, note that the representation of the Virasoro algebra has been inverted, then shifted by x^{-4}. Finally, while the Jacobi identity description highlights similarities between VOAs and VOCs, examining the corresponding "weak commutativity" properties reveals substantial differences (cf.* [FHL] *and* [Hub3]*). While these differences may seem in some sense unnatural, they are exactly the right differences to preserve the duality illuminated in the proof of Theorem 3.1.*

3. A family of examples of VOCs

One of the most natural questions to ask about vertex operator coalgebras is "what do they look like?", or even, "do any exist?". The main purpose of this paper is to answer the latter question in the affirmative and to provide concrete insight into the former question.

3.1. **A family of examples of VOCs.** Let the vector space $V = \coprod_{k\in\mathbb{Z}} V_{(k)}$ be a module over the Virasoro algebra, $\mathcal{V} = \oplus_{j\in\mathbb{Z}}\mathbb{F}L(j) \oplus \mathbb{F}d$, such that for all homogeneous vectors $L(0) \cdot v = \text{wt } (v)v$. We will say that a bilinear form $(\cdot, \cdot)$ on V is *Virasoro preserving* if it satisfies the condition

$$(L(k)v_1, v_2) = (v_1, L(-k)v_2) \tag{3.1}$$

for all $k \in \mathbb{Z}$, $v_1, v_2 \in V$. In particular, $k = 0$ in Property (3.1) indicates that all Virasoro preserving bilinear forms are graded, i.e.

$$(V_{(k)}, V_{(\ell)}) = 0$$

for $k \neq \ell$. If V is a VOA, the bilinear form is said to be *invariant* if, for all $u, v, w \in V$,

$$(Y(u,x)v, w) = (u, Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w). \tag{3.2}$$

Any invariant bilinear form on V is Virasoro preserving ((2.31) in [L]).

Note that there is a natural extension of $(\cdot,\cdot) : V^{\otimes 2} \to \mathbb{F}$ to $(\cdot,\cdot) : V^{\otimes 4} \to \mathbb{F}$ given by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2)$, for $u_1, u_2, v_1, v_2 \in V$.

Theorem 3.1. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra equipped with a nondegenerate, Virasoro preserving bilinear form $(\cdot,\cdot)$. Given the linear operators*

$$\begin{aligned} c : V &\to \mathbb{F} \\ v &\mapsto (v, \mathbf{1}), \end{aligned}$$

$$\begin{aligned} \rho : V &\to \mathbb{F} \\ v &\mapsto (v, \omega), \end{aligned}$$

and

$$\begin{aligned} \Lambda(x) : V &\to (V \otimes V)[[x, x^{-1}]] \\ v &\mapsto \Lambda(x)v = \sum_{k\in\mathbb{Z}} \Delta_k(v) x^{-k-1}, \end{aligned}$$

defined by

$$(\Lambda(x)u, v \otimes w) = (u, Y(v,x)w), \tag{3.3}$$

the quadruple (V, Λ, c, ρ) is a vertex operator coalgebra.

Proof. We will show that all 7 axioms for VOCs are satisfied.

1. Left counit: Given $u \in V$, for all $v \in V$

$$\begin{aligned} ((c \otimes Id_V)\Lambda(x)u, v) &= (\Lambda(x)u, \mathbf{1} \otimes v) \\ &= (u, Y(\mathbf{1}, x)v) \\ &= (u, v). \end{aligned}$$

Thus, by nondegeneracy, $(c \otimes Id_V)\Lambda(x)u = u$.

2. Cocreation: Given $u \in V$, then for all $v \in V$

$$\begin{aligned} ((Id_V \otimes c)\Lambda(x)u, v) &= (\Lambda(x)u, v \otimes \mathbf{1}) \\ &= (u, Y(v,x)\mathbf{1}) \\ &= (u, e^{xL(-1)}v) \in \mathbb{F}[x] \end{aligned}$$

and

$$\lim_{x\to 0}(u, e^{xL(-1)}v) = (u, v).$$

3. Truncation: Pick $N \in \mathbb{Z}$ such that $V_{(n)} = 0$ for all $n \leq N$. Given $u \in V_{(r)}$, let $v \in V_{(s)}$ and $w \in V_{(t)}$. Then we have

$$\begin{aligned}(\Lambda(x)u, v \otimes w) &= (u, Y(v, x)w)\\ &= \sum_{k\in\mathbb{Z}}(u, v_k w)x^{-k-1}.\end{aligned}$$

For $(u, v_k w)$ to be nonzero, we must have wt $u =$ wt $v +$ wt $w - k - 1$, i.e. $r = s+t-k-1$; but $s, t > N$, and thus we must have $r > 2N-k-1$ or $r-2N > -k-1$. Hence, $(\Lambda(x)u, v \otimes w) \in \mathbb{F}[[x^{-1}]]x^{r-2N-1}$ for any $s, t \in \mathbb{Z}$.

4. Jacobi identity: Given $u \in V$, then for all $v_1, v_2, v_3 \in V$

$$\begin{aligned}((Id_V \otimes \Lambda(x_2))\Lambda(x_1)u, v_1 \otimes v_2 \otimes v_3) &= (\Lambda(x_1)u, v_1 \otimes Y(v_2, x_2)v_3)\\ &= (u, Y(v_1, x_1)Y(v_2, x_2)v_3),\end{aligned} \tag{3.4}$$

$$\begin{aligned}((T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2)u,&v_1 \otimes v_2 \otimes v_3)\\ &= ((Id_V \otimes \Lambda(x_1))\Lambda(x_2)u, v_2 \otimes v_1 \otimes v_3)\\ &= (\Lambda(x_2)u, v_2 \otimes Y(v_1, x_1)v_3)\\ &= (u, Y(v_2, x_2)Y(v_1, x_1)v_3),\end{aligned} \tag{3.5}$$

$$\begin{aligned}((\Lambda(x_0) \otimes Id_V)\Lambda(x_2)u, v_1 \otimes v_2 \otimes v_3) &= (\Lambda(x_2)u, Y(v_1, x_0)v_2 \otimes v_3)\\ &= (u, Y(Y(v_1, x_0)v_2, x_2)v_3).\end{aligned} \tag{3.6}$$

Equations (3.4), (3.5) and (3.6) make it clear that the VOA Jacobi identity (2.11) is equivalent to the VOC Jacobi identity (2.4).

5. Virasoro algebra: Given $u \in V$, for all $v \in V$

$$\begin{aligned}((\rho \otimes Id_V)\Lambda(x)u, v) &= (\Lambda(x)u, \omega \otimes v)\\ &= (u, Y(\omega, x)v)\\ &= \sum_{k\in\mathbb{Z}}(u, L(k)v)x^{-k-2}\\ &= \sum_{j\in\mathbb{Z}}(L(j)u, v)x^{j-2}.\end{aligned} \tag{3.7}$$

Note that in the last equality we have used Virasoro preservation, (3.1).

Equation (3.7) shows that the Virasoro algebra bracket,

$$[L(j), L(k)] = (j-k)L(j+k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}d,$$

follows from the Virasoro bracket relation on VOAs.

6. Grading: Equation (3.7) shows that $L(0) = Res_x xY(\omega, x)$ so grading follows from VOAs.

7. $L(1)$-Derivative: Given $u \in V$, for all $v, w \in V$

$$\begin{aligned}((L(1)\otimes Id_V)\Lambda(x)u, v\otimes w) &= (\Lambda(x)u, L(-1)v\otimes w)\\ &= (u, Y(L(-1)v,x)w)\\ &= \frac{d}{dx}(u, Y(v,x)w)\\ &= \frac{d}{dx}(\Lambda(x)u, v\otimes w).\end{aligned}$$

Here the first equality uses Virasoro preservation. □

Li showed in [L] that if a simple VOA satisfies the condition $L(1)V_{(1)} = 0$ then there exists a nondegenerate, invariant bilinear form on V. Thus we are guaranteed a family of VOAs equipped with the type of form required for Theorem 3.1. Additionally, Heisenberg VOAs may be explicitly equipped with an appropriate bilinear form and they will be the focus of more concrete discussion in the next section.

Remark 3.2. *If preserving the underlying vector space and the Virasoro action is not a prerequisite, the proof of Theorem 3.1 shows that the graded dual space V' of any VOA V may be endowed with a VOC structure by replacing the bilinear form $(\cdot,\cdot)$ with the natural pairing $<\cdot,\cdot>$ of V' and V. This was noticed in collaboration with Katrina Barron.*

3.2. **Vertex operator algebras and coalgebras associated with Heisenberg algebras.** While the construction in the last section does describe a family of VOCs, it is not extremely explicit in nature. In this section we will explicitly construct VOCs from Heisenberg algebras. We begin with the construction of the vector space for the Heisenberg VOA following [D].

Let $\mathbf{h}$ be a finite dimensional vector space over $\mathbb{F}$ equipped with a symmetric, nondegenerate bilinear form $\langle\cdot,\cdot\rangle$. Since $\mathbf{h}$ may be considered as an abelian Lie algebra, let $\hat{\mathbf{h}}$ be the corresponding affine Lie algebra, i.e., let

$$\hat{\mathbf{h}} = \mathbf{h}\otimes\mathbb{F}[t,t^{-1}]\oplus\mathbb{F}c,$$

where c is nonzero, with the Lie bracket defined by

$$[\alpha\otimes t^m, \beta\otimes t^n] = \langle\alpha,\beta\rangle m\delta_{m,-n}c,$$
$$[\hat{\mathbf{h}}, c] = 0$$

for $\alpha,\beta\in\mathbf{h}$, $m,n\in\mathbb{Z}$. There is a natural $\mathbb{Z}$-grading on $\hat{\mathbf{h}}$ under which $\alpha\otimes t^m$ has weight $-m$ for all $\alpha\in\mathbf{h}$ and $m\in\mathbb{Z}$, and c has weight 0. The element $\alpha\otimes t^m$ of $\hat{\mathbf{h}}$ is usually denoted $\alpha(m)$. Three graded subalgebras are of interest:

$$\hat{\mathbf{h}}^+ = \mathbf{h}\otimes t\mathbb{F}[t],$$

$$\hat{\mathbf{h}}^- = \mathbf{h}\otimes t^{-1}\mathbb{F}[t^{-1}],$$

$$\hat{\mathbf{h}}_{\mathbb{Z}} = \hat{\mathbf{h}}^+\oplus\hat{\mathbf{h}}^-\oplus\mathbb{F}c.$$

The subalgebra $\hat{\mathbf{h}}_{\mathbb{Z}}$ is a *Heisenberg algebra*, by which we mean that its center is one-dimensional and is equal to its commutator subalgebra. Note that $\hat{\mathbf{h}}^+$ and $\hat{\mathbf{h}}^-$ are abelian, but that $\hat{\mathbf{h}}_{\mathbb{Z}}$ is necessarily non-abelian.

We consider the induced $\hat{\mathbf{h}}_{\mathbb{Z}}$-module

$$M(1) = U(\hat{\mathbf{h}}_{\mathbb{Z}}) \otimes_{U(\hat{\mathbf{h}}^+ \oplus \mathbb{F}c)} \mathbb{F}$$

where U indicates the universal enveloping algebra and $\mathbb{F}$ is viewed as a $\mathbb{Z}$-graded $(\hat{\mathbf{h}}^+ \oplus \mathbb{F}c)$-module by

$$\begin{aligned} c \cdot 1 &= 1, \\ \hat{\mathbf{h}}^+ \cdot 1 &= 0, \\ \deg 1 &= 0. \end{aligned}$$

The module $M(1)$ may be generalized (cf. [D] and [FLM]). $M(1)$ is linearly isomorphic to $S(\hat{\mathbf{h}}^-)$ in a way that preserves grading. Thus we often write basis elements of $M(1)$ as

$$v = \alpha_1(-n_1) \cdots \alpha_r(-n_r)$$

for $\alpha_i \in \mathbf{h}$, $n_i \in \mathbb{Z}_+$, $i = 1, \ldots, r$, and observe that v has weight $n_1 + \cdots + n_r$. (Tensor products are suppressed in this notation.) Note that the $\alpha_i(-n_i)$'s all commute so their order is irrelevant. Using the form $\langle \cdot, \cdot \rangle$ on $\mathbf{h}$, we may choose $\{\gamma_i\}_{i=1}^d$ to be an orthonormal basis and we lose no generality by considering only α_i from this set of basis elements. Thus, we will typically prove results for the set of generating elements

$$\mathrm{gen}M = \left\{\alpha_1(-n_1) \cdots \alpha_r(-n_r) | r \in \mathbb{N},\ \alpha_j \in \{\gamma_i\}_{i=1}^d,\ n_j \in \mathbb{Z}_+,\ j = 1, \ldots, r\right\}$$

and then extend linearly to all of $M(1)$.

Next we define a bilinear form on $M(1)$ which we will use throughout the rest of this section.

Lemma 3.3. *There is a unique bilinear form $(\cdot, \cdot)$ on $M(1)$ satisfying*

$$(\alpha(m) \cdot u, v) = (u, \alpha(-m) \cdot v), \tag{3.8}$$

$$(1, 1) = 1. \tag{3.9}$$

for all $u, v \in M(1)$, $\alpha \in \mathbf{h}$, $m \in \mathbb{Z} \setminus \{0\}$.

More precisely, given $v = \alpha_1(-n_1) \cdots \alpha_r(-n_r)$ let

$$p(v) = \alpha_r(n_r) \cdots \alpha_1(n_1)\alpha_1(-n_1) \cdots \alpha_r(-n_r) \in \mathbb{Z}_+ \subset M(1).$$

The unique bilinear form $(\cdot, \cdot)$ on $M(1)$ satisfying (3.8) and (3.9) is defined on basis elements $u, v \in genM$ by

$$(u, v) = \begin{cases} p(u) & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

Further, this form is nondegenerate, graded and symmetric.

Proof. We will construct the form in (3.10) from (3.8) and (3.9), thus showing the form is unique. Consider $u, v \in \mathrm{gen}M$ such that $u \neq v$. Then there is an element $\alpha(-n) \in \hat{\mathbf{h}}$ and a positive integer t such that $\alpha(-n)^t$ is contained in u or v but not in the other. We may assume $\alpha(-n)^t$ is in u, say $u = \alpha(-n)^t\alpha_1(-n_1)\cdots\alpha_r(-n_r)$. But then

$$\begin{aligned}(u,v) &= (\alpha(-n)^t\alpha_1(-n_1)\cdots\alpha_r(-n_r), v)\\ &= (\alpha_1(-n_1)\cdots\alpha_r(-n_r), \alpha(n)^t v)\\ &= (\alpha_1(-n_1)\cdots\alpha_r(-n_r), 0)\\ &= 0.\end{aligned}\tag{3.11}$$

(As a biproduct, (3.11) shows the form is graded and symmetric.) Now we need only examine the form applied to a single basis element. Let $u = \alpha_1(-n_1)\cdots\alpha_r(-n_r)$. Then

$$\begin{aligned}(u,u) &= (\alpha_1(-n_1)\cdots\alpha_r(-n_r), \alpha_1(-n_1)\cdots\alpha_r(-n_r))\\ &= (\alpha_r(n_r)\cdots\alpha_1(n_1)\alpha_1(-n_1)\cdots\alpha_r(-n_r), 1)\\ &= (p(u), 1)\\ &= p(u),\end{aligned}\tag{3.12}$$

thus proving that (3.10) is the unique form satisfying (3.8) and (3.9). Nondegeneracy is immediate from (3.12). □

Given $\alpha \in \mathbf{h}$, we will need the following three series in a formal variable, x, with coefficients in $\hat{\mathbf{h}}_{\mathbb{Z}}$:

$$\alpha^+(x) = \sum_{k\in\mathbb{Z}_-} \alpha(-k)x^{k-1},$$
$$\alpha^-(x) = \sum_{k\in\mathbb{Z}_+} \alpha(k)x^{-k-1},$$
$$\alpha(x) = \alpha^+(x) + \alpha^-(x).$$

The vertex operator algebra associated to a Heisenberg algebra is described using a normal ordering procedure, indicated by open colons $\circ\atop\circ$ $\circ\atop\circ$, which reorders the enclosed expression so that all operators $\alpha_1(-m)$ are placed to the left (multiplicatively) of all operators $\alpha_2(n)$, for $\alpha_1, \alpha_2 \in \mathbf{h}$, $m, n \in \mathbb{Z}_+$. For example,

$$\begin{aligned}{}^\circ_\circ\alpha_1(x)\alpha_2(x)^\circ_\circ v &= {}^\circ_\circ(\alpha_1^+(x) + \alpha_1^-(x))\alpha_2(x)^\circ_\circ v\\ &= \alpha_1^+(x)\alpha_2(x)v + \alpha_2(x)\alpha_1^-(x)v.\end{aligned}$$

This normal ordering moves degree-lowering operators to the right. Notice that $\alpha^-(x)$ will only produce elements of lower weight than the basis element of $M(1)$ to which it is applied, while $\alpha^+(x)$ will only produce elements of higher weight. Therefore applying all the degree-lowering operators before all the degree-raising operators guarantees that no single weight-space has infinitely many summands in it.

We now have the notation to describe the VOA associated to a Heisenberg algebra. Define a linear map $Y(\cdot,x) : M(1) \to (\mathrm{End}\ M(1))[[x,x^{-1}]]$ by

$$Y(v,x) = {\scriptstyle\circ\atop\scriptstyle\circ} \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dx}\right)^{n_1-1} \alpha_1(x) \right) \cdots \left(\frac{1}{(n_r-1)!} \left(\frac{d}{dx}\right)^{n_r-1} \alpha_r(x) \right) {\scriptstyle\circ\atop\scriptstyle\circ}$$

for $v = \alpha_1(-n_1)\cdots\alpha_r(-n_r)$. We also define two distinguished elements of $M(1)$, $\mathbf{1} = 1$ and $\omega = \frac{1}{2}\sum_{i=1}^{d} \gamma_i(-1)^2$, where $\{\gamma_i\}_{i=1}^{d}$ is the orthonormal basis of $\mathbf{h}$ as above.

Proposition 3.4. *The quadruple* $(M(1), Y, \mathbf{1}, \omega)$ *as defined above is a vertex operator algebra.*

Our treatment here largely mirrors [D], with the above proposition being Proposition 3.1 in [D]. A proof may be found in [G] or [LL]. For our purposes, however, the nondegenerate bilinear form on $M(1)$, described in (3.10), is equally relevant. We will now prove an additional property of that bilinear form.

Lemma 3.5. *The bilinear form on* $M(1)$ *defined in Equation (3.10) is Virasoro preserving.*

Proof. First, we will explicitly calculate the $L(k)$ operators and then show that $(L(k)v, w) = (v, L(-k)w)$. Symmetry of the form allows us to only consider $k \in \mathbb{N}$.

Using the definition of the $L(k)$ operators we see that $\sum_{k\in\mathbb{Z}} L(k)x^{-k-2} = Y(\frac{1}{2}\sum_{i=1}^{d}\gamma_i(-1)^2, x)$. Employing the definition of Y, for $k \in \mathbb{Z}_+$ we have

$$L(k) = \sum_{i=1}^{d} \left(\frac{1}{2}\sum_{j=1}^{k-1} \gamma_i(j)\gamma_i(k-j) + \sum_{j\in\mathbb{Z}_+} \gamma_i(-j)\gamma_i(k+j) \right)$$

$$L(-k) = \sum_{i=1}^{d} \left(\frac{1}{2}\sum_{j=1}^{k-1} \gamma_i(-j)\gamma_i(-k+j) + \sum_{j\in\mathbb{Z}_+} \gamma_i(-k-j)\gamma_i(j) \right).$$

Given $u, v \in \mathrm{gen}M$,

$$(L(0)u, v) = \mathrm{wt}\ (u)\ (u,v) = \mathrm{wt}\ (v)\ (u,v) = (u, L(0)v)$$

since either $(u,v) = 0$, or $u = v$ implying wt (u) = wt (v). For $k \in \mathbb{Z}_+$ we make use of (3.8) to observe that,

$$\begin{aligned}(L(k)u, v) &= \sum_{i=1}^{d} \left(\frac{1}{2}\sum_{j=1}^{k-1} (\gamma_i(j)\gamma_i(k-j)u, v) + \sum_{j\in\mathbb{Z}_+} (\gamma_i(-j)\gamma_i(k+j)u, v) \right) \\ &= \sum_{i=1}^{d} \left(\frac{1}{2}\sum_{j=1}^{k-1} (u, \gamma_i(-k+j)\gamma_i(-j)v) + \sum_{j\in\mathbb{Z}_+} (u, \gamma_i(-k-j)\gamma_i(j)v) \right) \\ &= (u, L(-k)v).\end{aligned}$$

□

Using the Heisenberg VOA $M(1)$ along with the nondegenerate, Virasoro preserving bilinear form defined in (3.10), we will follow the construction of a VOC in Section 3.1. First, we define a linear map $c : V \to \mathbb{F}$ by

$$c(\mathbf{1}) = 1$$
$$c(\alpha_1(-n_1)\cdots\alpha_r(-n_r)) = 0$$
where $r \geq 1$ and $\alpha_1(-n_1)\cdots\alpha_r(-n_r) \in \text{gen}M$. It is clear that $c(v) = (v, \mathbf{1})$ for all $v \in M(1)$. Next, we define a linear map $\rho : V \to \mathbb{F}$ by

$$\rho(\gamma_i(-1)^2) = 1$$

for each basis element γ_i of $\mathbf{h}$ and $\rho(v) = 0$ for v any other basis element of $M(1)$. Again it is clear that $\rho(v) = (v, \omega)$ for all $v \in M(1)$. Finally, we need to define a linear map $\mathit{\Lambda}(x) : V \to (V \otimes V)[[x, x^{-1}]]$ such that

$$(\mathit{\Lambda}(x)u, v \otimes w) = (u, Y(v, x)w).$$

For notational simplicity, given $\alpha \in \mathbf{h}$, $n \in \mathbb{Z}_+$ and x a formal variable, let

$$\alpha^+(n, x) = \sum_{k \in \mathbb{Z}_+} \binom{k-1}{n-1} \alpha(-k) x^{k-n}$$
$$\alpha^-(n, x) = \sum_{k \in \mathbb{Z}_+} \binom{-k-1}{n-1} \alpha(k) x^{-k-n}$$
$$\alpha(n, x) = \alpha^+(n, x) + \alpha^-(n, x)$$

so that

$$Y(\alpha_1(-n_1)\cdots\alpha_r(-n_r), x) = {}^{\circ}_{\circ}\alpha_1(n_1, x)\cdots\alpha_r(n_r, x){}^{\circ}_{\circ}.$$

It is also notationally useful to define

$$\alpha_{\bullet}^+(n, x) = \sum_{k \in \mathbb{Z}_+} \binom{-k-1}{n-1} \alpha(-k) x^{-k-n}$$
$$\alpha_{\bullet}^-(n, x) = \sum_{k \in \mathbb{Z}_+} \binom{k-1}{n-1} \alpha(k) x^{k-n}$$
$$\alpha_{\bullet}(n, x) = \alpha_{\bullet}^+(n, x) + \alpha_{\bullet}^-(n, x)$$

so that, by (3.8),

$$(v_1, \alpha^-(n, x)v_2) = (\alpha_{\bullet}^+(n, x)v_1, v_2), \tag{3.13}$$
$$(v_1, \alpha^+(n, x)v_2) = (\alpha_{\bullet}^-(n, x)v_1, v_2), \tag{3.14}$$
$$(v_1, \alpha(n, x)v_2) = (\alpha_{\bullet}(n, x)v_1, v_2). \tag{3.15}$$

Proposition 3.6. *Let v denote the basis element $\beta_1(-m_1)\cdots\beta_s(-m_s)$ and define $\mathit{\Lambda}(x) : V \to (V \otimes V)[[x, x^{-1}]]$ as*

$$\mathit{\Lambda}(x)u = \sum_{v \in genM} \frac{1}{p(v)} v \otimes {}^{\circ}_{\circ}\beta_{1\bullet}(m_1, x)\cdots\beta_{s\bullet}(m_s, x){}^{\circ}_{\circ}u.$$

For all $u, v, w \in M(1)$, $(\mathit{\Lambda}(x)u, v \otimes w) = (u, Y(v, x)w)$.

Proof. First, note that our definition of Λ is equivalent to

$$(\Lambda(x)u, v\otimes w) = (\frac{1}{p(v)}v\otimes \,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x)\,{}^\circ_\circ u, v\otimes w)$$

for all $u=\alpha_1(-n_1)\cdots\alpha_r(-n_r)$, $v=\beta_1(-m_1)\cdots\beta_s(-m_s)$, $w=\mu_1(-\ell_1)\cdots\mu_t(-\ell_t)$. Given these basis elements we use induction on s to show that

$$(\,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x)\,{}^\circ_\circ u, w) = (u,\,{}^\circ_\circ\beta_1(m_1,x)\cdots\beta_s(m_s,x)\,{}^\circ_\circ w). \tag{3.16}$$

For $s=0$, this is trivial. If we assume that (3.16) is true for $s-1$ and appeal to (3.13) and (3.14), we see that

$$(\,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x)\,{}^\circ_\circ u, w)$$

$$\begin{aligned}= (\beta^+_{s\bullet}(m_s,x)\,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s-1\bullet}(m_{s-1},x)\,{}^\circ_\circ u, w)\\ + (\,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s-1\bullet}(m_{s-1},x)\,{}^\circ_\circ\beta^-_{s\bullet}(m_s,x)u, w)\end{aligned}$$

$$\begin{aligned}= (\,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s-1\bullet}(m_{s-1},x)\,{}^\circ_\circ u, \beta^-_s(m_s,x)w)\\ + (\beta^-_{s\bullet}(m_s,x)u,\,{}^\circ_\circ\beta_1(m_1,x)\cdots\beta_{s-1}(m_{s-1},x)\,{}^\circ_\circ w)\end{aligned}$$

$$\begin{aligned}= (u,\,{}^\circ_\circ\beta_1(m_1,x)\cdots\beta_{s-1}(m_{s-1},x){}^\circ_\circ\,\beta^-_s(m_s,x)w)\\ + (u, \beta^+_s(m_s,x)\,{}^\circ_\circ\beta_1(m_1,x)\cdots\beta_{s-1}(m_{s-1},x)\,{}^\circ_\circ w)\end{aligned}$$

$$= (u,\,{}^\circ_\circ\beta_1(m_1,x)\cdots\beta_s(m_s,x)\,{}^\circ_\circ w).$$

Finally, using Equation (3.16) we see that

$$\begin{aligned}(\Lambda(x)u, v\otimes w) &= (\frac{1}{p(v)}v\otimes \,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x)\,{}^\circ_\circ u, v\otimes w)\\ &= \frac{(v,v)}{p(v)}(\,{}^\circ_\circ\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x)\,{}^\circ_\circ u, w)\\ &= (u,\,{}^\circ_\circ\beta_1(m_1,x)\cdots\beta_s(m_s,x)\,{}^\circ_\circ w)\\ &= (u, Y(v,x)w).\end{aligned}$$

□

Theorem 3.1 proves that the quadruple $(M(1), \Lambda, c, \rho)$ associated to the Heisenberg algebra $\hat{\mathbf{h}}_{\mathbb{Z}}$ is a vertex operator coalgebra.

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