Success in this course and in your future mathematics courses will require a good understanding of the basic properties of the real number system. Thus this book begins with a review of the real numbers. This chapter has been labeled “Chapter 0” to emphasize its review nature.

The first section of this chapter starts with the construction of the real line. This section contains as an optional highlight the ancient Greek proof that no rational number has a square equal to 2. This beautiful result appears here not because you will need it for calculus, but because it should be seen by everyone at least once.

Although this chapter will be mostly review, a thorough grounding in the real number system will serve you well throughout this course and then for the rest of your life. You will need good algebraic manipulation skills; thus the second section of this chapter reviews the fundamental algebra of the real numbers. You will also need to feel comfortable working with inequalities and absolute values, which are reviewed in the last section of this chapter.

Even if your instructor decides to skip this chapter, you may want to read through it. Make sure you can do all the exercises.
The Real Numbers

0.1 The Real Line

SECTION OBJECTIVES

By the end of this section you should

- understand the correspondence between the system of real numbers and the real line;
- appreciate the proof that no rational number has a square equal to 2.

The integers are the numbers

\[ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \]

where the dots indicate that the numbers continue without end in each direction. The sum, difference, and product of any two integers are also integers.

The quotient of two integers is not necessarily an integer. Thus we extend arithmetic to the rational numbers, which are numbers of the form

\[ \frac{m}{n}, \]

where \( m \) and \( n \) are integers and \( n \neq 0 \).

Division is the inverse of multiplication, in the sense that we want the equation

\[ \frac{m}{n} \cdot n = m \]

to hold. In the equation above, if we take \( n = 0 \) and (for example) \( m = 1 \), we get the nonsensical equation \( \frac{1}{0} \cdot 0 = 1 \). This equation is nonsensical because multiplying anything by 0 should give 0, not 1. To get around this problem, we leave expressions such as \( \frac{1}{0} \) undefined. In other words, division by 0 is prohibited.

The rational numbers form a terrifically useful system. We can add, multiply, subtract, and divide rational numbers (with the exception of division by 0) and stay within the system of rational numbers. Rational numbers suffice for all actual physical measurements, such as length and weight, of any desired accuracy.

However, geometry, algebra, and calculus force us to consider an even richer system of numbers—the real numbers. To see why we need to go beyond the rational numbers, we will investigate the real number line.

Construction of the Real Line

Imagine a horizontal line, extending without end in both directions. Pick an arbitrary point on this line and label it 0. Pick another arbitrary point to the right of 0 and label it 1, as in the figure below.
Two key points on the real line.

Once the points 0 and 1 have been chosen on the line, everything else is
determined by thinking of the distance between 0 and 1 as one unit of length.
For example, 2 is one unit to the right of 1, and then 3 is one unit to the right
of 2, and so on. The negative integers correspond to moving to the left of 0.
Thus \(-1\) is one unit to the left of 0, and then \(-2\) is one unit to the left of \(-1\),
and so on.

Integers on the real line.

If \(n\) is a positive integer, then \(\frac{1}{n}\) is to the right of 0 by the length obtained
by dividing the segment from 0 to 1 into \(n\) segments of equal length. Then
\(\frac{2}{n}\) is to the right of \(\frac{1}{n}\) by the same length, and \(\frac{3}{n}\) is to the right of \(\frac{2}{n}\)
by the same length again, and so on. The negative rational numbers are placed on
the line in a similar fashion, but to the left of 0.

In this fashion, we associate with every rational number a point on the
line. No figure can show the labels of all the rational numbers, because we
can include only finitely many labels. The figure below shows the line with
labels attached to a few of the points corresponding to rational numbers.

Some rational numbers on the real line.

We will use the intuitive notion that the line has no gaps and that every
conceivable distance can be represented by a point on the line. With these
concepts in mind, we call the line shown above the **real line**. We think of each
point on the real line as corresponding to a **real number**. The undefined
intuitive notions (such as “no gaps”) will become more precise when you
reach more advanced mathematics courses. For now, we let our intuitive
notions of the real line serve to define the system of real numbers.

**Is Every Real Number Rational?**

We have seen that every rational number corresponds to some point on the
real line. Does every point on the real line correspond to some rational num-
ber? In other words, is every real number rational?

If more and more labels of rational numbers were placed on the figure
above, the real line would look increasingly cluttered. Probably the first peo-
ple to ponder these issues thought that the rational numbers fill up the entire
real line. However, the ancient Greeks realized that this is not true. To see
how they came to this conclusion, we make a brief detour into geometry.
Recall that for a right triangle, the sum of the squares of the lengths of the two sides that form the right angle equals the square of the length of the hypotenuse. The figure below illustrates this result, which is called the Pythagorean Theorem.

This theorem is named in honor of the Greek mathematician and philosopher Pythagoras who proved it over 2500 years ago. The Babylonians discovered this result a thousand years earlier than that.

The Pythagorean Theorem for right triangles: $c^2 = a^2 + b^2$.

Now consider the special case where both sides that form the right angle have length 1, as in the figure below. In this case, the Pythagorean Theorem states that the length $c$ of the hypotenuse has a square equal to 2.

An isosceles right triangle. The Pythagorean Theorem implies that $c^2 = 2$.

Because we have constructed a line segment whose length $c$ satisfies the equation $c^2 = 2$, a point on the real line corresponds to $c$. In other words, there is a real number whose square equals 2. This raises the question of whether there exists a rational number whose square equals 2.

We could try to find a rational number whose square equals 2 by experimentation. One striking example is 

$$
\left(\frac{99}{70}\right)^2 = \frac{9801}{4900};
$$

here the numerator of the right side misses being twice the denominator by only 1. Although $\left(\frac{99}{70}\right)^2$ is close to 2, it is not exactly equal to 2.

Another example is $\frac{9369319}{6623109}$. The square of this rational number is approximately 1.9999999999992, which is very close to 2 but again is not exactly what we seek.

Because we have found rational numbers whose squares are very close to 2, you might suspect that with further cleverness we could find a rational number whose square equals 2. However, the ancient Greeks proved this is impossible. This course does not focus much on proofs, and probably your calculus course will not be proof oriented either. The Greek proof that there is no rational number whose square equals 2 could be skipped without endangering your future success. However, the Greek proof, as one of the
great intellectual achievements of humanity, should be experienced by every educated person. Thus it is presented below for your enrichment.

What follows is a proof by contradiction. We will start by assuming that the desired result is false. Using that assumption, we will arrive at a contradiction. So our assumption that the desired result was false must have been wrong. Thus the desired result is true.

Understanding the logical pattern of thinking that goes into this proof will be a valuable asset if you continue to other parts of mathematics beyond calculus.

No rational number has a square equal to 2.

Proof: Suppose there exist integers $m$ and $n$ such that

$$\left(\frac{m}{n}\right)^2 = 2.$$ 

By canceling any common factors, we can choose $m$ and $n$ to have no factors in common. In other words, $\frac{m}{n}$ is reduced to lowest terms.

The equation above is equivalent to the equation

$$m^2 = 2n^2.$$ 

This implies that $m^2$ is even; hence $m$ is even. Thus $m = 2k$ for some integer $k$. Substituting $2k$ for $m$ in the equation above gives

$$4k^2 = 2n^2,$$

or equivalently

$$2k^2 = n^2.$$ 

This implies that $n^2$ is even; hence $n$ is even.

We have now shown that both $m$ and $n$ are even, contradicting our choice of $m$ and $n$ as having no factors in common. This contradiction means our original assumption that there is a rational number whose square equals 2 must be false.

The result above shows that not every point on the real line corresponds to a rational number. In other words, not every real number is rational. Thus the following definition is useful:

Irrational numbers

A real number that is not rational is called an irrational number.
We have just seen that \( \sqrt{2} \), which is the positive real number whose square equals 2, is an irrational number. The real numbers \( \pi \) and \( e \), which we will encounter in later chapters, are also irrational numbers.

Once we have found one irrational number, finding others is much easier, as shown in the example below.

**Example 1**

Show that \( 3 + \sqrt{2} \) is an irrational number.

**Solution**
Suppose \( 3 + \sqrt{2} \) is a rational number. Because
\[
\sqrt{2} = (3 + \sqrt{2}) - 3,
\]
this implies that \( \sqrt{2} \) is the difference of two rational numbers, which implies that \( \sqrt{2} \) is a rational number, which is not true. Thus our assumption that \( 3 + \sqrt{2} \) is a rational number must be false. In other words, \( 3 + \sqrt{2} \) is an irrational number.

The next example provides another illustration of how to use one irrational number to generate another irrational number.

**Example 2**

Show that \( 8 \sqrt{2} \) is an irrational number.

**Solution**
Suppose \( 8 \sqrt{2} \) is a rational number. Because
\[
\sqrt{2} = \frac{8 \sqrt{2}}{8},
\]
this implies that \( \sqrt{2} \) is the quotient of two rational numbers, which implies that \( \sqrt{2} \) is a rational number, which is not true. Thus our assumption that \( 8 \sqrt{2} \) is a rational number must be false. In other words, \( 8 \sqrt{2} \) is an irrational number.

**Problems**

The problems in this section may be harder than typical problems found in the rest of this book.

1. Show that \( \frac{6}{7} + \sqrt{2} \) is an irrational number.
2. Show that \( 5 - \sqrt{2} \) is an irrational number.
3. Show that \( 3 \sqrt{2} \) is an irrational number.
4. Show that \( \frac{\sqrt{2}}{3} \) is an irrational number.
5. Show that \( 4 + 9 \sqrt{2} \) is an irrational number.
6. Explain why the sum of a rational number and an irrational number is an irrational number.
7. Explain why the product of a nonzero rational number and an irrational number is an irrational number.
8. Suppose \( t \) is an irrational number. Explain why \( \frac{1}{t} \) is also an irrational number.
9. Give an example of two irrational numbers whose sum is an irrational number.
10. Give an example of two irrational numbers whose sum is a rational number.
11. Give an example of three irrational numbers whose sum is a rational number.
12. Give an example of two irrational numbers whose product is an irrational number.
13. Give an example of two irrational numbers whose product is a rational number.
0.2 Algebra of the Real Numbers

SECTION OBJECTIVES
By the end of this section you should

- recall how to manipulate algebraic expressions using the commutative, associative, and distributive properties;
- understand the order of algebraic operations and the role of parentheses;
- recall the crucial algebraic identities involving additive inverses and multiplicative inverses.

The operations of addition, subtraction, multiplication, and division extend from the rational numbers to the real numbers. We can add, subtract, multiply, and divide any two real numbers and stay within the system of real numbers, again with the exception that division by 0 is prohibited.

In this section we review the basic algebraic properties of the real numbers. Because this material should indeed be review, no effort has been made to show how some of these properties follow from others. Instead, this section focuses on highlighting key properties that should become so familiar to you that you can use them comfortably and without effort.

Commutativity and Associativity

Commutativity is the formal name for the property stating that order does not matter in addition and multiplication:

\[ a + b = b + a \quad \text{and} \quad ab = ba \]

Here (and throughout this section) \( a, b, \) and other variables denote either arbitrary real numbers or expressions that take on values that are real numbers. For example, the commutativity of addition implies that \( x^2 + \frac{x}{3} = \frac{x}{3} + x^2. \)

Neither subtraction nor division is commutative because order does matter for those operations. For example, \( 5 - 3 \neq 3 - 5, \) and \( \frac{6}{2} \neq \frac{2}{6}. \)

Associativity is the formal name for the property stating that grouping does not matter in addition and multiplication:

\[ (a + b) + c = a + (b + c) \quad \text{and} \quad (ab)c = a(bc) \]

Expressions inside parentheses should be calculated before further computation. For example, \((a + b) + c\) should be calculated by first adding \( a \) and \( b, \) and then adding that sum to \( c. \) The associative property of addition

Exercises woven throughout this book have been designed to sharpen your algebraic manipulation skills as we cover other topics.
asserts that this number will be the same as \( a + (b + c) \), which should be calculated by first adding \( b \) and \( c \), and then adding that sum to \( a \).

Because of the associative property of addition, we can dispense with parentheses when adding three or more numbers, writing expressions such as

\[ a + b + c + d \]

without worrying about how the terms are grouped. Similarly, because of the associative property of multiplication we do not need parentheses when multiplying together three or more numbers. Thus we can write expressions such as \( abcd \) without specifying the order of multiplication or the grouping.

Neither subtraction nor division is associative because the grouping does matter for those operations. For example, \((9 - 6) - 2 = 3 - 2 = 1\), but \(9 - (6 - 2) = 9 - 4 = 5\), which shows that subtraction is not associative.

Because subtraction is not associative, we need a way to evaluate expressions that are written without parentheses. The standard practice is to evaluate subtractions from left to right unless parentheses indicate otherwise. For example, \(9 - 6 - 2\) should be interpreted to mean \((9 - 6) - 2\), which equals 1.

**The Order of Algebraic Operations**

Consider the expression

\[ 2 + 3 \cdot 7. \]

This expression contains no parentheses to guide us to which operation should be performed first. Should we first add 2 and 3, and then multiply the result by 7? If so, we would interpret the expression above as

\[ (2 + 3) \cdot 7, \]

which equals 35.

Or to evaluate

\[ 2 + 3 \cdot 7 \]

should we first multiply together 3 and 7, and then add 2 to that result. If so, we would interpret the expression above as

\[ 2 + (3 \cdot 7), \]

which equals 23.

So does \( 2 + 3 \cdot 7 \) equal \( (2 + 3) \cdot 7 \) or \( 2 + (3 \cdot 7) \)? The answer to this question depends on custom rather than anything inherent in the mathematical situation. Every mathematically literate person would interpret \( 2 + 3 \cdot 7 \) to mean \( 2 + (3 \cdot 7) \). In other words, people in the modern era have adopted the convention that multiplications should be performed before additions unless parentheses dictate otherwise. You need to become accustomed to this...
convention, which will be used throughout this course and all your further courses that use mathematics.

**Multiplication and division before addition and subtraction**

Unless parentheses indicate otherwise, products and quotients are calculated before sums and differences.

Thus, for example, \( a + bc \) is interpreted to mean \( a + (bc) \), although almost always we dispense with the parentheses and just write \( a + bc \).

As another illustration of the principle above, consider the expression

\[ 4m + 3n + 11(p + q). \]

The correct interpretation of this expression is that 4 should be multiplied by \( m \), 3 should be multiplied by \( n \), 11 should be multiplied by \( p + q \), and then the three numbers \( 4m, 3n, \) and \( 11(p + q) \) should be added together. In other words, the expression above equals

\[ (4m) + (3n) + (11(p + q)). \]

The three newly added sets of parentheses in the expression above are unnecessary, although it is not incorrect to include them. However, the version of the same expression without the unnecessary parentheses is cleaner and easier to read.

When parentheses are enclosed within parentheses, expressions in the innermost parentheses are evaluated first.

**Evaluate inner parentheses first**

In an expression with parentheses inside parentheses, evaluate the innermost parentheses first and then work outward.

Evaluate the expression \( 2(6 + 3(1 + 4)). \)

**Solution** Here the innermost parentheses surround \( 1 + 4 \). Thus start by evaluating that expression:

\[ 2\left(6 + 3 \cdot \frac{1+4}{5}\right) = 2(6 + 3 \cdot 5). \]

Now to evaluate the expression \( 6 + 3 \cdot 5 \), first evaluate \( 3 \cdot 5 \), getting 15, then add that to 6, getting 21. Multiplying by 2 completes our evaluation of this expression:

\[ 2 \cdot \frac{6 + 3 \cdot \frac{1+4}{5}}{21} = 42. \]
The Distributive Property

The distributive property connects addition and multiplication, converting a product with a sum into a sum of two products.

Distributive property

\[ a(b + c) = ab + ac \]

Because multiplication is commutative, the distributive property can also be written in the alternative form

\[ (a + b)c = ac + bc. \]

Sometimes you will need to use the distributive property to transform an expression of the form \( a(b + c) \) into \( ab + ac \), and sometimes you will need to use the distributive property in the opposite direction, transforming an expression of the form \( ab + ac \) into \( a(b + c) \). Because the distributive property is usually used to simplify an expression, the direction of the transformation depends on the context. The next example shows the use of the distributive property in both directions.

**Example 2**

Simplify the expression \( 2(3m + x) + 5x \).

**Solution**

First use the distributive property to transform \( 2(3m + x) \) into \( 6m + 2x \):

\[ 2(3m + x) + 5x = 6m + 2x + 5x. \]

Now use the distributive property again, but in the other direction, to transform \( 2x + 5x \) to \( (2 + 5)x \):

\[ 6m + 2x + 5x = 6m + (2 + 5)x = 6m + 7x. \]

Putting all this together, we have used the distributive property (twice) to transform \( 2(3m + x) + 5x \) into the simpler expression \( 6m + 7x \).

One of the most common algebraic manipulations involves expanding a product of sums, as in the following example.

**Example 3**

Expand \( (a + b)(c + d) \).
SOLUTION Think of $(c + d)$ as a single number and then apply the distributive property to the expression above, getting

$$(a + b)(c + d) = a(c + d) + b(c + d).$$

Now apply the distributive property twice more, getting

$$(a + b)(c + d) = ac + ad + bc + bd.$$ 

If you are comfortable with the distributive property, there is no need to memorize the last formula from the example above, because you can always derive it again. Furthermore, by understanding how the identity above was obtained, you should have no trouble finding formulas for more complicated expressions such as $(a + b)(c + d + t)$.

An important special case of the identity above occurs when $c = a$ and $d = b$. In that case we have

$$(a + b)(a + b) = a^2 + ab + ba + b^2,$$

which, with a standard use of commutativity, becomes the identity

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Additive Inverses and Subtraction

The additive inverse of a real number $a$ is the number $-a$ such that

$$a + (-a) = 0.$$

The connection between subtraction and additive inverses is captured by the identity

$$a - b = a + (-b).$$

In fact, the equation above can be taken as the definition of subtraction.

You need to be comfortable using the following identities that involve additive inverses and subtraction:

\begin{align*}
-(-a) &= a \\
-(a + b) &= -a - b \\
(-a)(-b) &= ab \\
(-a)b &= a(-b) = -(ab) \\
(a - b)c &= ac - bc \\
a(b - c) &= ab - ac
\end{align*}
Be sure to distribute the minus signs correctly when using the distributive property, as in the example below.

**EXAMPLE 4** Expand \((a + b)(a - b)\).

**SOLUTION** Start by thinking of \((a + b)\) as a single number and applying the distributive property. Then apply the distributive property twice more, paying careful attention to the minus signs:

\[(a + b)(a - b) = (a + b)a - (a + b)b\]
\[= a^2 + ba - ab - b^2\]
\[= a^2 - b^2\]

You need to become sufficiently comfortable with the following identities so that you can use them with ease.

**Identities arising from the distributive property**

\[(a + b)^2 = a^2 + 2ab + b^2\]
\[(a - b)^2 = a^2 - 2ab + b^2\]
\[(a + b)(a - b) = a^2 - b^2\]

**EXAMPLE 5** Without using a calculator, evaluate \(43 \times 37\).

**SOLUTION**  \(43 \times 37 = (40 + 3)(40 - 3) = 40^2 - 3^2 = 1600 - 9 = 1591\)

**Multiplicative Inverses and Division**

The multiplicative inverse of a real number \(b \neq 0\) is the number \(\frac{1}{b}\) such that

\[b \cdot \frac{1}{b} = 1.\]

The connection between division and multiplicative inverses is captured by the identity

\[\frac{a}{b} = a \cdot \frac{1}{b}.\]

In fact, the equation above can be taken as the definition of division. You need to be comfortable using the following identities that involve multiplicative inverses and division:
Let’s look at these identities a bit more carefully. In all the identities above, we assume that none of the denominators equals 0. The first identity above gives a formula for adding two fractions. The second identity above states that the product of two fractions can be computed by multiplying together the numerators and multiplying together the denominators. Note that the formula for adding fractions is more complicated than the formula for multiplying fractions.

The third identity above, when used to transform \( \frac{ac}{ad} \) into \( \frac{c}{d} \), is the usual simplification of canceling a common factor from the numerator and denominator. When used in the other direction to transform \( \frac{c}{d} \) into \( \frac{ac}{ad} \), the third identity above becomes the familiar procedure of multiplying the numerator and denominator by the same factor.

In the fourth identity above, the size of the fraction bars are used to indicate that

\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}
\]

should be interpreted to mean \( \frac{a}{b} \cdot \frac{d}{c} \). This identity gives the key to unraveling fractions that involve fractions, as shown in the following example.

**Example 6**

Simplify the expression

\[
\frac{\frac{a}{b}}{\frac{c}{d}}.
\]

**Solution** The size of the fraction bars indicates that the expression to be simplified is \( (a/b)/(c/d) \). Dividing by \( \frac{c}{d} \) is the same as multiplying by \( \frac{d}{c} \). Thus we have

\[
\frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.
\]
EXERCISES

For Exercises 1–4, determine how many different values can arise by inserting one pair of parentheses into the given expression.

1. \(19 - 12 - 8 - 2\)
2. \(3 - 7 - 9 - 5\)
3. \(6 + 3 \cdot 4 + 5 \cdot 2\)
4. \(5 \cdot 3 \cdot 2 + 6 \cdot 4\)

For Exercises 5–18, expand the given expression.

5. \((x - y)(z + w - t)\)
6. \((x + y - r)(z + w - t)\)
7. \((2x + 3)^2\)
8. \((3b + 5)^2\)
9. \((2c - 7)^2\)
10. \((4a - 5)^2\)
11. \((x + y + z)^2\)
12. \((x - 5y - 3z)^2\)
13. \((x + 1)(x - 2)(x + 3)\)
14. \((y - 2)(y - 3)(y + 5)\)
15. \((a + 2)(a - 2)(a^2 + 4)\)
16. \((b - 3)(b + 3)(b^2 + 9)\)
17. \(x^2(x + y)(\frac{x}{2} - \frac{1}{y})\)
18. \(a^2z(z - a)(\frac{1}{x} + \frac{1}{y})\)

For Exercises 19–40, simplify the given expression as much as possible.

19. \(4(2m + 3n) + 7m\)
20. \(3(2m + 4(n + 5p)) + 6n\)
21. \(\frac{3}{4} + \frac{6}{7}\)
22. \(\frac{2}{5} + \frac{7}{8}\)
23. \(\frac{3}{4} \cdot \frac{14}{39}\)
24. \(\frac{2}{3} \cdot \frac{15}{22}\)
25. \(\frac{5}{3}\)
26. \(\frac{6}{5}\)
27. \(\frac{m + 1}{2} + \frac{3}{n}\)
28. \(\frac{m}{3} + \frac{5}{n - 2}\)
29. \(\frac{2}{3} \cdot \frac{4}{5} + \frac{3}{4} \cdot 2\)
30. \(\frac{3}{5} \cdot \frac{2}{7} + \frac{5}{4} \cdot 2\)
31. \(\frac{2}{5} \cdot \frac{m + 3}{7} + \frac{1}{2}\)
32. \(\frac{3}{4} \cdot \frac{n - 2}{5} + \frac{7}{3}\)
33. \(\frac{2}{x + 3} + \frac{y - 4}{5}\)
34. \(\frac{x - 3}{4} - \frac{5}{y + 2}\)
35. \(\frac{1}{x - y} \left(\frac{x}{y} - \frac{y}{x}\right)\)
36. \(\frac{1}{y} \left(\frac{1}{x - y} - \frac{1}{x + y}\right)\)
37. \(\frac{1}{2} \cdot \frac{m + 3}{7} + \frac{1}{2}\)
38. \(\frac{1}{3} \cdot \frac{m + 3}{7} + \frac{1}{2}\)
39. \(\frac{1}{3} \cdot \frac{m + 3}{7} + \frac{1}{2}\)
40. \(\frac{2}{3} \cdot \frac{4}{5} + \frac{3}{4} \cdot 2\)

PROBLEMS

Some problems require considerably more thought than the exercises. Unlike exercises, problems usually have more than one correct answer.

41. Explain how you could show that \(51 \times 49 = 2499\) in your head by using the identity \((a + b)(a - b) = a^2 - b^2\).

42. Show that \(a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)\).

43. Give an example to show that division does not satisfy the associative property.

44. The sales tax in San Francisco is 8.5%. Diners in San Francisco often compute a 17% tip on their before-tax restaurant bill by simply doubling the sales tax. For example, a $64 dollar food and drink bill would come with a sales tax of $5.44; doubling that amount would lead to a 17% tip of $10.88 (which might be rounded up to $11). Explain why this technique is an application of the associativity of multiplication.
45. A quick way to compute a 15% tip on a restaurant bill is first to compute 10% of the bill (by shifting the decimal point) and then add half of that amount for the total tip. For example, 15% of a $43 restaurant bill is $4.30 + $2.15, which equals $6.45. Explain why this technique is an application of the distributive property.

46. The first letters of the phrase “Please excuse my dear Aunt Sally” are used by some people to remember the order of operations: parentheses, exponentiation (which we will discuss in a later chapter), multiplication, division, addition, subtraction. Make up a catchy phrase that serves the same purpose but with exponentiation excluded.

47. (a) Verify that
\[
\frac{16}{2} - \frac{25}{5} = \frac{16 - 25}{2 - 5}.
\]
(b) From the example above you may be tempted to think that
\[
\frac{a}{b} - \frac{c}{d} = \frac{a - c}{b - d}
\]
provided none of the denominators equals 0. Give an example to show that this is not true.

WORKED-OUT SOLUTIONS to Odd-numbered Exercises

Do not read these worked-out solutions before first struggling to do the exercises yourself. Otherwise you risk the danger of mimicking the techniques shown here without understanding the ideas.

For Exercises 1–4, determine how many different values can arise by inserting one pair of parentheses into the given expression.

1. \(19 - 12 - 8 - 2\)

**SOLUTION** Here are the possibilities:
\[
19(-12 - 8 - 2) = -418 \\
19(-12 - 8) - 2 = -382 \\
19(-12) - 8 - 2 = -238 \\
(19 - 12) - 8 - 2 = -3 \\
19 - 12 - (8 - 2) = 1 \\
19 - (12 - 8) - 2 = 13 \\
19 - (12 - 8) - 2 = 17 \\
19 - 12 - 8(-2) = 23 \\
19 - 12(-8) - 2 = 113 \\
19 - 12(-8 - 2) = 139
\]

Thus ten values are possible; they are \(-418, -382, -238, -3, 1, 13, 17, 23, 113, \text{ and } 139\).

3. \(6 + 3 \cdot 4 + 5 \cdot 2\)

**SOLUTION** Here are the possibilities:
\[
(6 + 3 \cdot 4 + 5 \cdot 2) = 28 \\
6 + (3 \cdot 4 + 5) \cdot 2 = 40 \\
(6 + 3) \cdot 4 + 5 \cdot 2 = 46 \\
6 + 3 \cdot (4 + 5) \cdot 2 = 48 \\
6 + 3 \cdot (4 + 5) \cdot 2 = 60
\]

Other possible ways to insert one pair of parentheses lead to values already included in the list above. For example,
\[
(6 + 3 \cdot 4 + 5) \cdot 2 = 46.
\]
Thus five values are possible; they are 28, 40, 46, 48, and 60.

For Exercises 5–18, expand the given expression.

5. \((x - y)(z + w - t)\)

**SOLUTION**

\[(x - y)(z + w - t) = x(z + w - t) - y(z + w - t)
= xz + xw - xt - yz - yw + yt\]

7. \((2x + 3)^2\)

**SOLUTION**

\[(2x + 3)^2 = (2x)^2 + 2 \cdot (2x) \cdot 3 + 3^2
= 4x^2 + 12x + 9\]

9. \((2c - 7)^2\)

**SOLUTION**

\[(2c - 7)^2 = (2c)^2 - 2 \cdot (2c) \cdot 7 + 7^2
= 4c^2 - 28c + 49\]

11. \((x + y + z)^2\)

**SOLUTION**

\[(x + y + z)^2
= (x + y + z)(x + y + z)
= x^2 + xy + xz + yx + y^2 + yz
+ zx + zy + z^2
= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz\]

13. \((x + 1)(x - 2)(x + 3)\)

**SOLUTION**

\[(x + 1)(x - 2)(x + 3)
= ((x + 1)(x - 2))(x + 3)
= (x^2 - 2x + x - 2)(x + 3)
= (x^2 - x - 2)(x + 3)
= x^3 + 3x^2 - x^2 - 3x - 2x - 6
= x^3 + 2x^2 - 5x - 6\]

15. \((a + 2)(a - 2)(a^2 + 4)\)

**SOLUTION**

\[(a + 2)(a - 2)(a^2 + 4) = ((a + 2)(a - 2))(a^2 + 4)
= (a^2 - 4)(a^2 + 4)
= a^4 - 16\]

17. \(xy(x + y)(\frac{1}{x} - \frac{1}{y})\)

**SOLUTION**

\[xy(x + y)(\frac{1}{x} - \frac{1}{y})
= xy(x + y)(\frac{y - x}{xy})
= (x + y)(y - x)
= y^2 - x^2\]

For Exercises 19–40, simplify the given expression as much as possible.

19. \(4(2m + 3n) + 7m\)

**SOLUTION**

\[4(2m + 3n) + 7m
= 8m + 12n + 7m
= 15m + 12n\]

21. \(\frac{3}{4} + \frac{6}{7}\)

**SOLUTION**

\[\frac{3}{4} + \frac{6}{7} = \frac{3}{4} \cdot \frac{7}{7} + \frac{6}{4} \cdot \frac{4}{4}
= \frac{21}{28} + \frac{24}{28}
= \frac{45}{28}\]

23. \(\frac{3}{4} \cdot \frac{14}{39}\)
25. $\frac{3}{4} \cdot \frac{14}{39} = \frac{3 \cdot 14}{4 \cdot 39} = \frac{7}{2 \cdot 13} = \frac{7}{26}$

27. $\frac{m + 1}{2} + \frac{3}{n}$

SOLUTION

\[
\frac{m + 1}{2} + \frac{3}{n} = \frac{m + 1 \cdot n}{2n} + \frac{3 \cdot 2}{2n} = \frac{(m + 1)n + 3 \cdot 2}{2n} = \frac{mn + n + 6}{2n}
\]

29. $\frac{2}{3} \cdot \frac{4}{5} + \frac{3}{4} \cdot 2$

SOLUTION

\[
\frac{2}{3} \cdot \frac{4}{5} + 3 \cdot 2 = \frac{8}{15} + 6 = \frac{8}{15} + \frac{2 \cdot 3 \cdot 15}{15} = \frac{16 + 45}{30} = \frac{61}{30}
\]

31. $\frac{2}{5} \cdot \frac{m + 3}{7} + \frac{1}{2}$

SOLUTION

\[
\frac{2}{5} \cdot \frac{m + 3}{7} + 1 \cdot 2 = \frac{2m + 6}{35} + \frac{1 \cdot 2}{2} = \frac{2m + 6 + 35}{35} = \frac{4m + 42}{35} = \frac{4m + 12 + 30}{35} = \frac{4m + 42}{35} = \frac{4m + 12 + 30}{35} = \frac{4m + 47}{70} = \frac{4m + 47}{70}
\]

33. $\frac{2}{x + 3} + \frac{y - 4}{5}$
0.3 **Inequalities**

**SECTION OBJECTIVES**

By the end of this section you should
- recall the algebraic properties involving positive and negative numbers;
- understand inequalities;
- be able to use interval notation for the four types of intervals;
- be able to use interval notation involving $-\infty$ and $\infty$;
- be able to work with unions of intervals;
- be able to manipulate and interpret expressions involving absolute value.

**Positive and Negative Numbers**

A number is called **positive** if it is right of 0 on the real line.

A number is called **negative** if it is left of 0 on the real line.

Every number is either right of 0, left of 0, or equals 0. Thus every number is either positive, negative, or 0.

All of the following properties should already be familiar to you.

**Algebraic properties of positive and negative numbers**

- The sum of two positive numbers is positive.
- The sum of two negative numbers is negative.
- The additive inverse of a positive number is negative.
- The additive inverse of a negative number is positive.
- The product of two positive numbers is positive.
- The product of two negative numbers is positive.
- The product of a positive number and a negative number is negative.
- The multiplicative inverse of a positive number is positive.
- The multiplicative inverse of a negative number is negative.
Lesser and Greater

We say that a number \( a \) is less than a number \( b \), written \( a < b \), if \( a \) is left of \( b \) on the real line. Equivalently, \( a < b \) if and only if \( b - a \) is positive. In particular, \( b \) is positive if and only if \( 0 < b \).

\[
\begin{array}{c}
& a < b \\
\end{array}
\]

We say that \( a \) is less than or equal to \( b \), written \( a \leq b \), if \( a < b \) or \( a = b \).

Thus the statement \( x < 4 \) is true if \( x \) equals 3 but false if \( x \) equals 4, whereas the statement \( x \leq 4 \) is true if \( x \) equals 3 and also true if \( x \) equals 4.

We say that \( b \) is greater than \( a \), written \( b > a \), if \( b \) is right of \( a \) on the real line. Thus \( b > a \) means the same as \( a < b \). Similarly, we say that \( b \) is greater than or equal to \( a \), written \( b \geq a \), if \( b > a \) or \( b = a \). Thus \( b \geq a \) means the same as \( a \leq b \).

We now begin discussion of a series of simple but crucial properties of inequalities. The first property we will discuss is called transitivity.

Transitivity

If \( a < b \) and \( b < c \), then \( a < c \).

To see why transitivity holds, suppose \( a < b \) and \( b < c \). Then \( a \) is left of \( b \) on the real line and \( b \) is left of \( c \). This implies that \( a \) is left of \( c \), which means that \( a < c \); see the figure below.

\[
\begin{array}{c}
& a < b \quad \text{and} \quad b < c \\
\end{array}
\]

Transitivity: \( a < b \) and \( b < c \) implies that \( a < c \).

Often multiple inequalities are written together as a single string of inequalities. Thus \( a < b < c \) means the same thing as \( a < b \) and \( b < c \).

Our next result shows that we can add inequalities.

Addition of inequalities

If \( a < b \) and \( c < d \), then \( a + c < b + d \).

To see why this is true, note that if \( a < b \) and \( c < d \), then \( b - a \) and \( d - c \) are positive numbers. Because the sum of two positive numbers is positive, this implies that \( (b - a) + (d - c) \) is positive. In other words, \( (b + d) - (a + c) \) is positive. This means that \( a + c < b + d \), as desired.

The next result states that we can multiply both sides of an inequality by a positive number and preserve the inequality. However, if both sides of an inequality are multiplied by a negative number, then the direction of the inequality must be reversed.
Multiplication of an inequality

Suppose \( a < b \).

- If \( c > 0 \), then \( ac < bc \).
- If \( c < 0 \), then \( ac > bc \).

To see why this is true, first suppose \( c > 0 \). We are assuming that \( a < b \), which means that \( b - a \) is positive. Because the product of two positive numbers is positive, this implies that \((b - a)c\) is positive. In other words, \( bc - ac \) is positive, which means that \( ac < bc \), as desired.

Now consider the case where \( c < 0 \). We are still assuming that \( a < b \), which means that \( b - a \) is positive. Because the product of a positive number and a negative number is negative, this implies that \((b - a)c\) is negative. In other words, \( bc - ac \) is negative, which means that \( ac > bc \), as desired.

An important special case of the result above is obtained by setting \( c = -1 \), which gives the following result:

Additive inverse and inequalities

If \( a < b \), then \( -a > -b \).

In other words, the direction of an inequality must be reversed when taking additive inverses of both sides.

The next result shows that the direction of an inequality must also be reversed when taking multiplicative inverses of both sides, unless one side is negative and the other side is positive.

Multiplicative inverse and inequalities

Suppose \( a < b \).

- If \( a \) and \( b \) are both positive or both negative, then \( \frac{1}{a} > \frac{1}{b} \).
- If \( a < 0 < b \), then \( \frac{1}{a} < \frac{1}{b} \).

To see why this is true, first suppose \( a \) and \( b \) are both positive or both negative. In either case, \( ab \) is positive. Thus \( \frac{1}{ab} > 0 \). Thus we can multiply both sides of the inequality \( a < b \) by \( \frac{1}{ab} \), preserving the direction of the inequality. This gives

\[
a \cdot \frac{1}{ab} < b \cdot \frac{1}{ab},
\]

which is the same as \( \frac{1}{b} < \frac{1}{a} \), or equivalently \( \frac{1}{a} > \frac{1}{b} \), as desired.

The case where \( a < 0 < b \) is even easier. In this case \( \frac{1}{a} \) is negative and \( \frac{1}{b} \) is positive. Thus \( \frac{1}{a} < \frac{1}{b} \), as desired.
**Intervals**

We begin this subsection with an imprecise definition.

*Set*

A set is a collection of objects.

The collection of positive numbers is an example of a set, as is the collection of odd negative integers. Most of the sets considered in this book are collections of real numbers, which at least removes some of the vagueness from the word “objects”.

If a set contains only finitely many objects, then the objects in the set can be explicitly displayed between the symbols \{\}. For example, the set consisting of the numbers 4, \(-\frac{17}{7}\), and \(\sqrt{2}\) can be denoted by

\[\{4, \frac{-17}{7}, \sqrt{2}\}\].

Sets can also be denoted by a property that characterizes objects of the set. For example, the set of real numbers greater than 2 can be denoted by

\[\{x : x > 2\}\].

Here the notation \{x : \ldots\} should be read to mean “the set of real numbers \(x\) such that” and then whatever follows. There is no particular \(x\) here. The variable is simply a convenient device to describe a property, and the symbol used for the variable does not matter. Thus \(\{x : x > 2\}\) and \(\{y : y > 2\}\) and \(\{t : t > 2\}\) all denote the same set, which can also be described (without mentioning any variables) as the set of real numbers greater than 2.

A special type of set occurs so often in mathematics that it gets its own name, which is given by the following definition.

*Interval*

An interval is a set of real numbers that contains all numbers between any two numbers in the set.

For example, the set of positive numbers is an interval because all numbers between any two positive numbers are positive. As a nonexample, the set of integers is not an interval because 0 and 1 are in this set, but \(\frac{1}{2}\), which is between 0 and 1, is not in this set. As another nonexample, the set of rational numbers is not an interval, because 1 and 2 are in this set, but \(\sqrt{2}\), which is between 1 and 2, is not in this set.

Intervals are so useful in mathematics that special notation has been designed for them. Suppose \(a\) and \(b\) are numbers with \(a < b\). We define the following four intervals with endpoints \(a\) and \(b\):
The definition of $[a, b]$ also makes sense when $a = b$; the interval $[a, a]$ consists of the single number $a$.

The term “half-closed” would make as much sense as “half-open”.

Intervals

- The open interval $(a, b)$ with endpoints $a$ and $b$ is the set of numbers between $a$ and $b$, not including either endpoint:
  $$(a, b) = \{x : a < x < b\}.$$

- The closed interval $[a, b]$ with endpoints $a$ and $b$ is the set of numbers between $a$ and $b$, including both endpoints:
  $$[a, b] = \{x : a \leq x \leq b\}.$$

- The half-open interval $[a, b)$ with endpoints $a$ and $b$ is the set of numbers between $a$ and $b$, including $a$ but not including $b$:
  $$[a, b) = \{x : a \leq x < b\}.$$

- The half-open interval $(a, b]$ with endpoints $a$ and $b$ is the set of numbers between $a$ and $b$, including $b$ but not including $a$:
  $$(a, b] = \{x : a < x \leq b\}.$$

With this notation, a parenthesis indicates that the corresponding endpoint is not included in the set, and a straight bracket indicates that the corresponding endpoint is included in the set. Thus the interval $(3, 7]$ includes the numbers 4, $\sqrt{17}$, 5.49, and the endpoint 7 (along with many other numbers), but does not include the numbers 2 or 9 or the endpoint 3.

Sometimes we need to use intervals that extend arbitrarily far to the left or to the right on the real number line. Suppose $a$ is a real number. We define the following four intervals with endpoint $a$:

Intervals

- The interval $(a, \infty)$ is the set of numbers greater than $a$:
  $$(a, \infty) = \{x : x > a\}.$$

- The interval $[a, \infty)$ is the set of numbers greater than or equal to $a$:
  $$[a, \infty) = \{x : x \geq a\}.$$

- The interval $(-\infty, a)$ is the set of numbers less than $a$:
  $$(-\infty, a) = \{x : x < a\}.$$

- The interval $(-\infty, a]$ is the set of numbers less than or equal to $a$:
  $$(-\infty, a] = \{x : x \leq a\}.$$
Here the symbol $\infty$, called **infinity**, should be thought of simply as a notational convenience. Neither $\infty$ nor $-\infty$ is a real number; these symbols have no meaning in this context other than as notational shorthand. For example, the interval $(2, \infty)$ is defined to be the set of real numbers greater than 2 (note that $\infty$ is not mentioned in this definition). The notation $(2, \infty)$ is often used because writing $(2, \infty)$ is easier than writing $\{x : x > 2\}$.

As before, a parenthesis indicates that the corresponding endpoint is not included in the set, and a straight bracket indicates that the corresponding endpoint is included in the set. Thus the interval $(2, \infty)$ does not include the endpoint 2, but the interval $[2, \infty)$ does include the endpoint 2. Both of the intervals $(2, \infty)$ and $[2, \infty)$ include 2.5 and 98765 (along with many other numbers); neither of these intervals includes 1.5 or $-857$.

There do not exist intervals with a closed bracket adjacent to $-\infty$ or $\infty$. For example, $[-\infty, 2]$ and $[2, \infty)$ do not make sense because the closed brackets indicate that both endpoints should be included. The symbols $-\infty$ and $\infty$ can never be included in a set of real numbers because these symbols do not denote real numbers.

In later chapters we will occasionally find it useful to work with the union of two intervals. Here is the definition of union:

**Union**

The **union** of two sets $A$ and $B$, denoted $A \cup B$, is the set of objects that are contained in at least one of the sets $A$ and $B$.

Thus $A \cup B$ consists of the objects (usually numbers) that belong either to $A$ or to $B$ or to both $A$ and $B$.

---

**EXAMPLE 1**

Write $(1, 5) \cup (3, 7)$ as an interval.

**SOLUTION**

As can be seen from the figure here, every number in the interval $(1, 5]$ is either in $(1, 5)$ or is in $(3, 7]$ or is in both $(1, 5)$ and $(3, 7]$. The figure shows that $(1, 5) \cup (3, 7] = (1, 7]$.

The next example goes in the other direction, starting with a set and then writing it as a union of intervals.

**EXAMPLE 2**

Write the set of nonzero real numbers as the union of two intervals.

**SOLUTION**
The set of nonzero real numbers is the union of the set of negative numbers and the set of positive numbers. In other words, the set of nonzero real numbers equals $(-\infty, 0) \cup (0, \infty)$.
Absolute Value

The absolute value of a number is its distance from 0; here we are thinking of numbers as points on the real line. For example, the absolute value of \( \frac{3}{2} \) equals \( \frac{3}{2} \), as can be seen in the figure below. More interestingly, the absolute value of \( -\frac{3}{2} \) equals \( \frac{3}{2} \).

![Figure showing the real line with points -2, \(-\frac{3}{2}\), -1, 0, \(\frac{3}{2}\), and 2.](image)

The absolute value of a number is its distance to 0.

The absolute value of a number \( b \) is denoted by \( |b| \). Thus \( |\frac{3}{2}| = \frac{3}{2} \) and \( |-\frac{3}{2}| = \frac{3}{2} \). Here is the formal definition of absolute value:

**Absolute value**

The absolute value of a number \( b \), denoted \( |b| \), is defined by

\[
|b| = \begin{cases} 
  b & \text{if } b \geq 0 \\
  -b & \text{if } b < 0
\end{cases}
\]

For example, \( -\frac{3}{2} < 0 \), and thus by the formula above \( |-\frac{3}{2}| \) equals \(-\frac{3}{2}\), which equals \( \frac{3}{2} \).

The concept of absolute value is fairly simple—just strip away the minus sign from any number that happens to have one. However, this rule can be applied only to numbers, not to expressions whose value is unknown. For example, if we encounter the expression \( |- (x + y) | \), we cannot simplify this expression to \( x + y \) unless we know that \( x + y \geq 0 \). If \( x + y \) happens to be negative, then \( -(x + y) | = -(x + y) \); stripping away the negative sign would be incorrect in this case.

Inequalities involving absolute values can be written without using an absolute value, as shown in the following example.

**Example 3**

(a) Write the inequality \( |x| < 2 \) without using an absolute value.

(b) Write the set \( \{ x : |x| < 2 \} \) as an interval.

**Solution**

(a) A number has absolute value less than 2 if only and only if its distance from 0 is less than 2, and this happens if and only if the number is between \(-2 \) and \(2 \). Hence the inequality \( |x| < 2 \) could be written as

\[-2 < x < 2.\]

(b) The inequality above implies that the set \( \{ x : |x| < 2 \} \) equals the open interval \((-2, 2)\). 

\[\text{Stephen F Austin}\]
In the next example, we end up with an interval not centered at 0.

**EXAMPLE 4**

The set \( \{ x : |x - 5| < 1 \} \) is the set of points on the real line whose distance to 5 is less than 1.

(a) Write the inequality \(|x - 5| < 1\) without using an absolute value.

(b) Write the set \( \{ x : |x - 5| < 1 \} \) as an interval.

**SOLUTION**

(a) The absolute value of a number is less than 1 precisely when the number is between \(-1\) and 1. Thus the inequality \( |x - 5| < 1 \) is equivalent to

\[-1 < x - 5 < 1.\]

Adding 5 to all three parts of the inequality above transforms it to the inequality

\[4 < x < 6.\]

(b) The inequality above implies that the set \( \{ x : |x - 5| < 1 \} \) equals the open interval \((4, 6)\).

In the next example, we deal with a slightly more abstract situation, using symbols rather than specific numbers. You should begin to get comfortable working in such situations. To get a good understanding of an abstract piece of mathematics, start by looking at an example using concrete numbers, as in Example 4, before going on to a more abstract setting, as in Example 5.

**EXAMPLE 5**

The set \( \{ x : |x - b| < h \} \) is the set of points on the real line whose distance to \( b \) is less than \( h \).

Suppose \( b \) is a real number and \( h > 0 \).

(a) Write the inequality \(|x - b| < h\) without using an absolute value.

(b) Write the set \( \{ x : |x - b| < h \} \) as an interval.

**SOLUTION**

(a) The absolute value of a number is less than \( h \) precisely when the number is between \(-h\) and \( h \). Thus the inequality \(|x - b| < h\) is equivalent to

\[-h < x - b < h.\]

Adding \( b \) to all three parts of the inequality above transforms it to the inequality

\[b - h < x < b + h.\]

(b) The inequality above implies that the set \( \{ x : |x - b| < h \} \) equals the open interval \((b - h, b + h)\).

\[
\text{\{x : |x - b| < h\} is the open interval of length 2h centered at b.}
\]
Equations involving absolute values must often be solved by considering multiple possibilities. Here is a simple example:

**EXAMPLE 6**

Find all numbers \( t \) such that

\[ |3t - 4| = 10. \]

**SOLUTION**

The equation \(|3t - 4| = 10\) implies \(3t - 4 = 10\) or \(3t - 4 = -10\).

Solving these equations for \( t \) gives \( t = \frac{14}{3} \) or \( t = -3 \). Substituting these values for \( t \) back into the original equation shows that both \( \frac{14}{3} \) and \(-3\) are indeed solutions.

A more complicated example would ask for all numbers \( x \) such that

\[ |x - 3| + |x - 4| = 9. \]

To find the solutions to this equation, think of the set of real numbers as the union of the three intervals \((\infty, 3), [3, 4), \text{ and } [4, \infty)\) and consider what the equation above becomes for \( x \) in each of those three intervals.

**EXERCISES**

**In Exercises 1–6, find all numbers \( x \) satisfying the given equation.**

1. \(|2x - 6| = 11\)
2. \(|5x + 8| = 19\)
3. \(|\frac{x - 4}{x} - 2| = 2\)
4. \(|\frac{3x + 2}{x - 4}| = 5\)
5. \(|x - 3| + |x - 4| = 9\)
6. \(|x - 1| + |x - 2| = 7\)

**In Exercises 7–16, write each union as a single interval.**

7. \([2, 7) \cup [5, 20)\)
8. \([-8, -3) \cup [-6, -1)\)
9. \([-2, 8) \cup (-1, 4)\)
10. \((-9, -2) \cup [-7, -5]\)
11. \((3, \infty) \cup [2, 8]\)
12. \((-\infty, 4) \cup (-2, 6)\)
13. \((-\infty, -3) \cup [-5, \infty)\)
14. \((-\infty, -6) \cup (-8, 12)\)
15. \((-3, \infty) \cup [-5, \infty)\)
16. \((-\infty, -10] \cup (-\infty, -8]\)
17. Give four examples of pairs of real numbers \( a \) and \( b \) such that

\[ |a + b| = 2 \quad \text{and} \quad |a| + |b| = 8. \]

18. Give four examples of pairs of real numbers \( a \) and \( b \) such that

\[ |a + b| = 3 \quad \text{and} \quad |a| + |b| = 11. \]

**In Exercises 19–30, write each set as an interval or as a union of two intervals.**

19. \(\{x : |x - 4| < \frac{1}{10}\}\)
20. \(\{x : |x + 2| < \frac{1}{100}\}\)
21. \(\{x : |x + 4| < \frac{1}{10}; \text{ here } \varepsilon > 0\}\)
22. \(\{x : |x - 2| < \frac{1}{10}; \text{ here } \varepsilon > 0\}\)
23. \(\{y : |y - a| < \varepsilon; \text{ here } \varepsilon > 0\}\)
24. \(\{y : |y + b| < \varepsilon; \text{ here } \varepsilon > 0\}\)
25. \(\{x : |3x - 2| < \frac{1}{3}\}\)
26. \(\{x : |4x - 3| < \frac{1}{4}\}\)
27. \(\{x : |x| > 2\}\)
28. \(\{x : |x| > 9\}\)
29. \(\{x : |x - 5| \geq 3\}\)
30. \(\{x : |x + 6| \geq 2\}\)

**The intersection of two sets of numbers consists of all numbers that are in both sets. If \( A \) and \( B \) are sets, then their intersection is denoted by \( A \cap B \). In Exercises 31–40, write each intersection as a single interval.**

31. \([2, 7) \cap [5, 20)\)
32. \([-8, -3) \cap [-6, -1)\)
33. \([-2, 8) \cap (-1, 4)\)
34. \((-9, -2) \cap [-7, -5]\)
35. \((3, \infty) \cap [2, 8]\)
36. \((-\infty, 4) \cap (-2, 6)\)
37. \((-\infty, -3) \cap [-5, \infty)\)
38. \((-\infty, -6) \cap (-8, 12)\)
39. \((-3, \infty) \cap [-5, \infty)\)
40. \((-\infty, -10] \cap (-\infty, -8]\)
PROBLEMS

41. Suppose $a$ and $b$ are numbers. Explain why either $a < b$, $a = b$, or $a > b$.

42. Show that if $a < b$ and $c \leq d$, then $a + c < b + d$.

43. Show that if $b$ is a positive number and $a < b$, then \[ \frac{a}{b} < \frac{a + 1}{b + 1}. \]

44. In contrast to Problem 47 in Section 0.2, show that there do not exist positive numbers $a$, $b$, $c$, and $d$ such that \[ \frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}. \]

45. (a) True or false:
   
   If $a < b$ and $c < d$, then $c - b < d - a$.

   (b) Explain your answer to part (a). This means that if the answer to part (a) is "true", then you should explain why $c - b < d - a$ whenever $a < b$ and $c < d$; if the answer to part (a) is "false", then you should give an example of numbers $a$, $b$, $c$, and $d$ such that $a < b$ and $c < d$ but $c - b \geq d - a$.

46. (a) True or false:
   
   If $a < b$ and $c < d$, then $ac < bd$.

   (b) Explain your answer to part (a). This means that if the answer to part (a) is "true", then you should explain why \[ ac < bd \] whenever $a < b$ and $c < d$; if the answer to part (a) is "false", then you should give an example of numbers $a$, $b$, $c$, and $d$ such that $a < b$ and $c < d$ but $ac \geq bd$.

47. (a) True or false:
   
   If $0 < a < b$ and $0 < c < d$, then \[ \frac{a}{d} < \frac{b}{c}. \]

   (b) Explain your answer to part (a). This means that if the answer to part (a) is "true", then you should explain why \[ \frac{a}{d} < \frac{b}{c} \] whenever $0 < a < b$ and $0 < c < d$; if the answer to part (a) is "false", then you should give an example of numbers $a$, $b$, $c$, and $d$ such that $0 < a < b$ and $0 < c < d$ but \[ \frac{a}{d} \geq \frac{b}{c}. \]

48. Explain why every open interval containing 0 contains an open interval centered at 0.

49. Give an example of an open interval and a closed interval whose union equals the interval $(2, 5)$.

50. Give an example of an open interval and a closed interval whose intersection equals the interval $(2, 5)$.

51. Give an example of an open interval and a closed interval whose union equals the interval $[-3, 7]$.

52. Give an example of an open interval and a closed interval whose intersection equals the interval $[-3, 7]$.

53. Explain why the equation $|8x - 3| = -2$ has no solutions.

54. Explain why $|a^2| = a^2$ for every real number $a$.

55. Explain why $|ab| = |a||b|$ for all real numbers $a$ and $b$.

56. Explain why $|-a| = |a|$ for all real numbers $a$.

57. Explain why $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$ for all real numbers $a$ and $b$ (with $b \neq 0$).

58. (a) Show that if $a \geq 0$ and $b \geq 0$, then $|a + b| = |a| + |b|$.

   (b) Show that if $a \geq 0$ and $b < 0$, then $|a + b| \leq |a| + |b|$.

   (c) Show that if $a < 0$ and $b \geq 0$, then $|a + b| \leq |a| + |b|$.

   (d) Show that if $a < 0$ and $b < 0$, then $|a + b| = |a| + |b|$.

   (e) Explain why the previous four items imply that $|a + b| \leq |a| + |b|$ for all real numbers $a$ and $b$. 
59. Show that if \( a \) and \( b \) are real numbers such that
\[
|a + b| < |a| + |b|,
\]
then \( ab < 0 \).

60. Show that
\[
|a| - |b| \leq |a - b|
\]
for all real numbers \( a \) and \( b \).

**WORKED-OUT SOLUTIONS to Odd-numbered Exercises**

**In Exercises 1–6, find all numbers \( x \) satisfying the given equation.**

1. \(|2x - 6| = 11\)

**SOLUTION** The equation \(|2x - 6| = 11\) implies that \(2x - 6 = 11\) or \(2x - 6 = -11\). Solving these equations for \( x \) gives \( x = \frac{17}{2} \) or \( x = -\frac{5}{2} \).

3. \(|\frac{x+1}{x-1}| = 2\)

**SOLUTION** The equation \(|\frac{x+1}{x-1}| = 2\) implies that \(\frac{x+1}{x-1} = 2\) or \(\frac{x+1}{x-1} = -2\). Solving these equations for \( x \) gives \( x = 3 \) or \( x = \frac{1}{3} \).

5. \(|x - 3| + |x - 4| = 9\)

**SOLUTION** First, consider numbers \( x \) such that \( x \geq 4 \). In this case, we have \( x - 3 \geq 0 \) and \( x - 4 \geq 0 \), which implies that \( |x - 3| = x - 3 \) and \( |x - 4| = x - 4 \). Thus the original equation becomes
\[
x - 3 + x - 4 = 9,
\]
which can be rewritten as \(2x - 7 = 9\), which can easily be solved to yield \( x = 8 \). Substituting 8 for \( x \) in the original equation shows that \( x = 8 \) is indeed a solution (make sure you do this check).

Second, consider numbers \( x \) such that \( x < 3 \). In this case, we have \( x - 3 < 0 \) and \( x - 4 < 0 \), which implies that \(|x - 3| = 3 - x\) and \(|x - 4| = 4 - x\). Thus the original equation becomes
\[
3 - x + 4 - x = 9,
\]
which can be rewritten as \(7 - 2x = 9\), which can easily be solved to yield \( x = -1 \). Substituting \(-1\) for \( x \) in the original equation shows that \( x = -1 \) is indeed a solution (make sure you do this check).

Third, we need to consider the only remaining possibility, which is that \( 3 \leq x < 4 \). In this case, we have \( x - 3 \geq 0 \) and \( x - 4 < 0 \), which implies

**In Exercises 7–16, write each union as a single interval.**

7. \([2, 7) \cup [5, 20)\)

**SOLUTION** The first interval is the set \( \{x : 2 \leq x < 7\} \), which includes the left endpoint 2 but does not include the right endpoint 7. The second interval is the set \( \{x : 5 \leq x < 20\} \), which includes the left endpoint 5 but does not include the right endpoint 20. The set of numbers that are in at least one of these sets equals \( \{x : 2 \leq x < 20\} \), as can be seen in the figure below:

Thus \([2, 7) \cup [5, 20) = [2, 20)\).

9. \([-2, 8] \cup (-1, 4)\)

**SOLUTION** The first interval is the set \( \{x : -2 \leq x \leq 8\} \), which includes both endpoints. The second interval is the set \( \{x : -1 < x < 4\} \), which does not include either endpoint. The set of numbers that are in at least one of these sets equals \( \{x : -2 \leq x \leq 8\} \), as can be seen in the following figure:
11. \((3, \infty) \cup [2, 8]\)

**SOLUTION** The first interval is the set \(\{x : 3 < x\}\), which does not include the left endpoint and which has no right endpoint. The second interval is the set \(\{x : 2 \leq x \leq 8\}\), which includes both endpoints. The set of numbers that are in at least one of these sets equals \(\{x : 2 \leq x\}\), as can be seen in the figure below:

Thus \((3, \infty) \cup [2, 8] = [2, \infty)\).

13. \((-\infty, -3) \cup [-5, \infty)\)

**SOLUTION** The first interval is the set \(\{x : x < -3\}\), which has no left endpoint and which does not include the right endpoint. The second interval is the set \(\{x : -5 \leq x\}\), which includes the left endpoint and which has no right endpoint. The set of numbers that are in at least one of these sets equals the entire real line, as can be seen in the figure below:

Thus \((-\infty, -3) \cup [-5, \infty) = (-\infty, \infty)\).

15. \((-3, \infty) \cup [-5, \infty)\)

**SOLUTION** The first interval is the set \(\{x : -3 < x\}\), which does not include the left endpoint and which has no right endpoint. The second interval is the set \(\{x : -5 \leq x\}\), which includes the left endpoint and which has no right endpoint. The set of numbers that are in at least one of these sets equals \(\{x : -5 \leq x\}\), as can be seen in the figure below:

Thus \((-3, \infty) \cup [-5, \infty) = [-5, \infty)\).

17. Give four examples of pairs of real numbers \(a\) and \(b\) such that \(|a + b| = 2\) and \(|a| + |b| = 8\).

**SOLUTION** First consider the case where \(a \geq 0\) and \(b \geq 0\). In this case, we have \(a + b \geq 0\). Thus the equations above become

\[a + b = 2\quad\text{and}\quad a + b = 8.\]

There are no solutions to the simultaneous equations above, because \(a + b\) cannot simultaneously equal both 2 and 8.

Next consider the case where \(a < 0\) and \(b < 0\). In this case, we have \(a + b < 0\). Thus the equations above become

\[-a - b = 2\quad\text{and}\quad -a - b = 8.\]

There are no solutions to the simultaneous equations above, because \(-a - b\) cannot simultaneously equal both 2 and 8.

Now consider the case where \(a \geq 0\), \(b < 0\), and \(a + b \geq 0\). In this case the equations above become

\[a + b = 2\quad\text{and}\quad a - b = 8.\]

Solving these equations for \(a\) and \(b\), we get \(a = 5\) and \(b = -3\).

Now consider the case where \(a \geq 0\), \(b < 0\), and \(a + b < 0\). In this case the equations above become

\[-a - b = 2\quad\text{and}\quad a - b = 8.\]

Solving these equations for \(a\) and \(b\), we get \(a = 3\) and \(b = -5\).

Now consider the case where \(a < 0\), \(b \geq 0\), and \(a + b \geq 0\). In this case the equations above become

\[a + b = 2\quad\text{and}\quad -a + b = 8.\]
Solving these equations for \(a\) and \(b\), we get 
\[ a = -3 \quad \text{and} \quad b = 5. \]
Now consider the case where \(a < 0, b \geq 0\), and 
\[ a + b < 0. \] 
In this case the equations above become 
\[ -a - b = 2 \quad \text{and} \quad -a + b = 8. \]
Solving these equations for \(a\) and \(b\), we get 
\[ a = -5 \quad \text{and} \quad b = 3. \]
At this point, we have considered all possible cases. Thus the only solutions are 
\(a = 5, b = -3\), or \(a = 3, b = -5\), or \(a = -3, b = 5\), or 
\(a = -5, b = 3\).

**In Exercises 19–30, write each set as an interval 
or as a union of two intervals.**

19. \(\{x : |x - 4| < \frac{1}{10}\}\)

**SOLUTION** The inequality \(|x - 4| < \frac{1}{10}\) is equivalent to the inequality 
\[ -\frac{1}{10} < x - 4 < \frac{1}{10}. \]
Add 4 to all parts of this inequality, getting 
\[ 4 - \frac{1}{10} < x < 4 + \frac{1}{10}, \]
which is the same as 
\[ \frac{39}{10} < x < \frac{41}{10}. \]
Thus \(\{x : |x - 4| < \frac{1}{10}\} = (\frac{39}{10}, \frac{41}{10}).\)

21. \(\{x : |x + 4| < \frac{5}{2}\}\); here \(\varepsilon > 0\)

**SOLUTION** The inequality \(|x + 4| < \frac{5}{2}\) is equivalent to the inequality 
\[ -\frac{5}{2} < x + 4 < \frac{5}{2}. \]
Add \(-4\) to all parts of this inequality, getting 
\[ -4 - \frac{5}{2} < x < -4 + \frac{5}{2}. \]
Thus \(\{x : |x + 4| < \frac{5}{2}\} = (-4 - \frac{5}{2}, -4 + \frac{5}{2}).\)

23. \(\{y : |y - a| < \varepsilon\}\); here \(\varepsilon > 0\)

**SOLUTION** The inequality \(|y - a| < \varepsilon\) is equivalent to the inequality 
\[ -\varepsilon < y - a < \varepsilon. \]
Add \(a\) to all parts of this inequality, getting 
\[ a - \varepsilon < y < a + \varepsilon. \]
Thus \(\{y : |y - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon).\)

25. \(\{x : |3x - 2| < \frac{1}{4}\}\)

**SOLUTION** The inequality \(|3x - 2| < \frac{1}{4}\) is equivalent to the inequality 
\[ -\frac{1}{4} < 3x - 2 < \frac{1}{4}. \]
Add 2 to all parts of this inequality, getting 
\[ \frac{7}{4} < 3x < \frac{9}{4}. \]
Now divide all parts of this inequality by 3, getting 
\[ \frac{7}{12} < x < \frac{3}{4}. \]
Thus \(\{x : |3x - 2| < \frac{1}{4}\} = (\frac{7}{12}, \frac{3}{4}).\)

27. \(\{x : |x| > 2\}\)

**SOLUTION** The inequality \(|x| > 2\) means that \(x > 2\) or \(x < -2\). Thus 
\(\{x : |x| > 2\} = (-\infty, -2) \cup (2, \infty).\)

29. \(\{x : |x - 5| \geq 3\}\)

**SOLUTION** The inequality \(|x - 5| \geq 3\) means that \(x - 5 \geq 3\) or \(x - 5 \leq -3\). Adding \(5\) to both sides of these equalities shows that \(x \geq 8\) or \(x \leq 2\). Thus \(\{x : |x - 5| \geq 3\} = (-\infty, 2] \cup [8, \infty).\)

The intersection of two sets of numbers consists of all numbers that are in both sets. If \(A\) and \(B\) are sets, then their intersection is denoted by \(A \cap B\). In Exercises 31–40, write each intersection as a single interval.

31. \((2, 7) \cap [5, 20]\)

**SOLUTION** The first interval is the set \(\{x : 2 \leq x < 7\}\), which includes the left endpoint 2 but does not include the right endpoint 7. The second interval is the set \(\{x : 5 \leq x < 20\}\), which includes the left endpoint 5 but does not include the right endpoint 20. The set of numbers that are in both these sets equals \(\{x : 5 \leq x < 7\}\), as can be seen in the figure below:

```
2 7

\[\frac{3}{2} \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19 \quad 20\]
```
Thus \([2, 7) \cap [5, 20) = [5, 7)\).

33. \([-2, 8] \cap (-1, 4)\)

**SOLUTION** The first interval is the set \(\{x : -2 \leq x \leq 8\}\), which includes both endpoints. The second interval is the set \(\{x : -1 < x < 4\}\), which includes neither endpoint. The set of numbers that are in both these sets equals \(\{x : -1 < x < 4\}\), as can be seen in the figure below:

\[
\begin{align*}
-2 & \quad 8 \\
-1 & \quad 4
\end{align*}
\]

Thus \([-2, 8] \cap (-1, 4) = (-1, 4)\).

35. \((3, \infty) \cap [2, 8)\)

**SOLUTION** The first interval is the set \(\{x : 3 < x\}\), which does not include the left endpoint and which has no right endpoint. The second interval is the set \(\{x : 2 \leq x \leq 8\}\), which includes both endpoints. The set of numbers that are in both these sets equals \(\{x : 2 < x \leq 8\}\), as can be seen in the figure below:

\[
\begin{align*}
3 & \quad \infty \\
2 & \quad 8
\end{align*}
\]

Thus \((3, \infty) \cap [2, 8) = (3, 8)\).

37. \((-\infty, -3) \cap [-5, \infty)\)

**SOLUTION** The first interval is the set \(\{x : x < -3\}\), which has no left endpoint and which does not include the right endpoint. The second interval is the set \(\{x : -5 \leq x\}\), which includes the left endpoint and which has no right endpoint. The set of numbers that are in both these sets equals \(\{x : -5 \leq x < -3\}\), as can be seen in the figure below:

\[
\begin{align*}
-\infty & \quad -3 \\
-5 & \quad \infty
\end{align*}
\]

Thus \((-\infty, -3) \cap [-5, \infty) = [-5, -3)\).

39. \((-3, \infty) \cap [-5, \infty)\)

**SOLUTION** The first interval is the set \(\{x : x < -3\}\), which does not include the left endpoint and which has no right endpoint. The second interval is the set \(\{x : -5 \leq x\}\), which includes the left endpoint and which has no right endpoint. The set of numbers that are in both these sets equals \(\{x : -3 < x\}\), as can be seen in the figure below:

\[
\begin{align*}
-3 & \quad \infty \\
-5 & \quad \infty
\end{align*}
\]

Thus \((-3, \infty) \cap [-5, \infty) = (-3, \infty)\).
CHAPTER SUMMARY

To check that you have mastered the most important concepts and skills covered in this chapter, make sure that you can do each item in the following list:

- Explain the correspondence between the system of real numbers and the real line.
- Simplify algebraic expressions using the commutative, associative, and distributive properties.
- List the order of algebraic operations.
- Explain how parentheses are used to alter the order of algebraic operations.
- Use the algebraic identities involving additive inverses and multiplicative inverses.
- Manipulate inequalities.
- Use interval notation for open intervals, closed intervals, and half-open intervals.
- Use interval notation involving $-\infty$ and $\infty$, with the understanding that $-\infty$ and $\infty$ are not real numbers.
- Write inequalities involving an absolute value without using an absolute value.
- Compute the union of intervals.

To review a chapter, go through the list above to find items that you do not know how to do, then reread the material in the chapter about those items. Then try to answer the chapter review questions below without looking back at the chapter.

CHAPTER REVIEW QUESTIONS

1. Explain how the points on the real line correspond to the set of real numbers.

2. Show that $7 - 6\sqrt{2}$ is an irrational number.

3. What is the commutative property for addition?

4. What is the commutative property for multiplication?

5. What is the associative property for addition?

6. What is the associative property for multiplication?

7. Expand $(t + w)^2$.

8. Expand $(u - v)^2$.

9. Expand $(x - y)(x + y)$.

10. Expand $(a + b)(x - y - z)$.

11. Expand $(a + b - c)^2$.

12. Simplify the expression $\frac{\frac{1}{x} - 1}{b}$.

13. Find all real numbers $x$ such that $|3x - 4| = 5$.

14. Give an example of two numbers $x$ and $y$ such that $|x + y|$ does not equal $|x| + |y|$.

15. Suppose $0 < a < b$ and $0 < c < d$. Explain why $ac < bd$.

16. Write the set $\{t : |t - 3| < \frac{1}{4}\}$ as an interval.

17. Write the set $\{w : |5w + 2| < \frac{1}{2}\}$ as an interval.

18. Explain why the sets $\{x : |8x - 5| < 2\}$ and $\{t : |5 - 8t| < 2\}$ are the same set.

19. Write $[-5, 6) \cup [-1, 9)$ as an interval.

20. Write $(-\infty, 4] \cup (3, 8]$ as an interval.

21. Explain why $[7, \infty]$ is not an interval of real numbers.

22. Write the set $\{t : |2t + 7| \geq 5\}$ as a union of two intervals.

23. Is the set of all real numbers $x$ such that $x^2 > 3$ an interval?