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The duality between vertex operator algebras and coalgebras, modules and comodules

Keith Hubbard

Abstract. We construct an equivalence between the categories of vertex operator algebras and vertex operator coalgebras. We then investigate to what degree weak modules, generalized modules and ordinary modules carry corresponding comodule structures, as well as when various comodules carry module structure.

1. Introduction

In this paper we describe a universal procedure for constructing an infinite family of multiplications on the dual space of a vertex operator algebra, and then show that this procedure yields a vertex operator coalgebra. This construction provides an equivalence of categories between the category of vertex operator algebras and the category of vertex operator coalgebras. Nearly all the axioms of vertex operator coalgebras are proven directly from the corresponding axiom of a vertex operator algebra. However, in the case of the truncation condition, such a parallelism does not exist. Using equivalent characterizations of vertex operator algebras and vertex operator coalgebras which refer to the weights of the operators involved, we are able to maintain the parallel correspondence in our proof. This transition to an equivalent characterization may seem insignificant but has implications when we shift our attention to modules. We begin by recalling the definition of weak modules, generalized modules and ordinary modules over a vertex operator algebra, then discuss the corresponding comodule notions over a vertex operator coalgebra. We then investigate whether a particular type of module structure induces a corresponding comodule structure. As may be suspected from the above discussion, the truncation condition plays a central role. Finally, we discuss how contragredient modules allow us to construct module and comodule structures on the same space in addition to constructing them on the dual space.

The motivation for these questions dates back to the early days of the development of the theory of vertex operator algebras (and vertex algebras). Vertex operator algebras arose in the 1980s in conformal field theory, with near simultaneous motivation coming from work in infinite-dimensional Lie algebras, finite sporadic

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simple groups and modular function theory [**BPZ, B, FLM, FHL**]. Modules over vertex operator algebras were almost immediately of vital importance in all of these areas. In new areas such as modular tensor categories, weakened versions of the module axioms became necessary [**HL2, DLM**].

Of vital importance to the field of vertex operator algebras are matters of symmetry and duality. Even in the 1980s, G. Segal wrote about the importance of duality of operators [**S**] and Frenkel, Huang and Lepowsky described a module structure on the dual space of a module [**FHL**]. Following the general principal of [**S**] and the rigorous construction of geometric vertex operator algebras in [**H2**], the notion of vertex operator coalgebras was introduced in [**Hub1**]. In [**H2**], motivated by the geometric interaction of $n \in \mathbb{N}$ closed strings combining in space-time to form one closed string, Huang defined geometric vertex operator algebras in the context of genus-zero Riemann surfaces with tubes up to conformal equivalence. He proved that the category of geometric vertex operator algebras is isomorphic to the category of vertex operator algebras. A similar procedure was used in [**Hub1, Hub2**]: motivated by the geometry of one closed string splitting into $n \in \mathbb{N}$ closed strings in space-time, the notion of geometric vertex operator coalgebra was introduced as well as the notion of vertex operator coalgebra. These two notions were shown to be isomorphic. Comodules over vertex operator coalgebras were defined in [**Hub4**] and algebraic notions of duality were discussed in [**Hub3**]. In this paper, we investigate the natural generalizations of comodule over a vertex operator coalgebra and delve into the above described questions of duality regarding vertex operator algebras, coalgebras, and their modules and comodules.

2. Vertex operator algebras and vertex operator coalgebras

2.1. Delta functions and expansions of formal variables. We define the "formal δ-function" to be

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

Given commuting formal variables x_1, x_2 and n an integer, $(x_1 \pm x_2)^n$ will be understood to be expanded in nonnegative integral powers of x_2. Note that the δ-function applied to $\frac{x_1 - x_2}{x_0}$, where x_0, x_1 and x_2 are commuting formal variables, is a formal power series in nonnegative integral powers of x_2 (cf. [**FLM**], [**FHL**]).

The following property of δ-functions will be relevant:

$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \tag{2.1}$$

This identity may be derived conceptually [**LL**] or by direct expansion.

2.2. Two equivalent notions of vertex operator algebra. We begin by recalling the definition of vertex operator algebra [VOA] and an equivalent characterization using the weights of particular operators rather than a 'truncation condition'. This characterization and an explicit enumeration of the axioms for a VOA will be necessary below.

Definition 2.1. A *vertex operator algebra (over* $\mathbb{C}$*) of rank* $d \in \mathbb{C}$ is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$

$$V = \coprod_{k \in \mathbb{Z}} V_{(k)},$$

such that $\dim V_{(k)} < \infty$ for $k \in \mathbb{Z}$ and $V_{(k)} = 0$ for k sufficiently small, together with a linear map $V \otimes V \to V[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} Y(\cdot, x) : V &\mapsto (\mathrm{End}\ V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{k\in\mathbb{Z}} v_k x^{-k-1} \qquad (\text{where } v_n \in \mathrm{End}\ V), \end{aligned}$$

equipped with two distinguished homogeneous vectors in V, $\mathbf{1}$ (the *vacuum*) and ω (the *Virasoro element*), satisfying the following 7 axioms:

1. Left unit: For all $v \in V$,

$$Y(\mathbf{1}, x)v = v. \tag{2.2}$$

2. Creation: For all $v \in V$,

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \tag{2.3}$$

$$\lim_{x\to 0} Y(v, x)\mathbf{1} = v. \tag{2.4}$$

3. Truncation: Given $v, w \in V$, then $v_k w = 0$ for k sufficiently large, or equivalently,

$$Y(v, x)w \in V((x)). \tag{2.5}$$

4. Jacobi Identity: For all $u, v \in V$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2). \end{aligned} \tag{2.6}$$

5. Virasoro Algebra: The Virasoro algebra bracket,

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}d, \tag{2.7}$$

holds for all $j, k \in \mathbb{Z}$, where

$$Y(\omega, x) = \sum_{k\in\mathbb{Z}} L(k)x^{-k-2}. \tag{2.8}$$

6. Grading: For each $k \in \mathbb{Z}$ and $v \in V_{(k)}$,

$$L(0)v = kv. \tag{2.9}$$

7. $L(-1)$-Derivative: Given $v \in V$,

$$\frac{d}{dx} Y(v, x) = Y(L(-1)v, x). \tag{2.10}$$

We denote a VOA either by V or by the quadruple $(V, Y, \mathbf{1}, \omega)$. A VOA *homomorphism* (or *morphism* in the category of VOAs) is a linear function f from one VOA U to another VOA V such that $f(\mathbf{1}) = \mathbf{1}$ and such that for all $u_1, u_2 \in U$,

$$f(Y(u_1, x)u_2) = Y(f(u_1), x)f(u_2).$$

A well known fact, dating back to [**B**], is

$$\text{wt } (v_k)\ w = r + s - k - 1 \qquad \text{for } v \in V_{(r)},\ w \in V_{(s)}. \tag{2.11}$$

It is usually expressed as saying that v_k is a weight $r-k-1$ operator since it raises every homogeneous subspace into the one that is $r-k-1$ above it. The following is a standard result (cf. [**FHL**]).

Proposition 2.2. The weight condition (2.11) is a consequence of the Jacobi identity (2.6), grading (2.9), and the $L(-1)$-derivative property (2.10).

Corollary 2.3. In the presence of the other axioms, the weight condition (2.11) may replace the truncation condition (2.5) in the definition of vertex operator algebra.

Proof. We have already established in Proposition 2.2 that (2.11) is a consequence of the standard definition of VOA sans truncation. Now assume (2.11) and let $v \in V_{(r)}$, $w \in V_{(s)}$. By positive energy, i.e. $V_{(n)} = 0$ for $n << 0$, if $u_k v$ is nonzero, there is a fixed $N <$ wt $u_k v = r + s - k - 1$. Then $N - r - s < -k - 1$, i.e. $Y(u, x)v \in V x^{N-r-s}[[x]]$. □

Remark 2.4. A stronger corollary, specifically, that (2.5) is redundant since it is proven from (2.6), (2.9), and (2.10), is untrue since the Jacobi identity (2.6) is not well defined without (2.5). However, it is well defined when (2.11) and positive energy are applied.

2.3. Two equivalent notions of vertex operator coalgebra. Vertex operator coalgebras [VOCs] are motivated by an isomorphism with geometric vertex operator coalgebras (which are built from correlation functions of geometrically defined objects associated to closed strings splitting in conformal field theory) [**Hub1, Hub2**] just as VOAs were tied to geometric vertex operator algebras by Huang in [**H1, H2**]. We recall the definition of VOC and discuss another weighting condition which is provably equivalent to the VOC truncation condition.

Definition 2.5. A *vertex operator coalgebra (over $\mathbb{C}$) of rank $d \in \mathbb{C}$* is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$

$$V = \coprod_{k \in \mathbb{Z}} V_{(k)}$$

such that $\dim V_{(k)} < \infty$ for $k \in \mathbb{Z}$ and $V_{(k)} = 0$ for k sufficiently small, together with linear maps

$$\begin{aligned} \curlyvee(x) : V &\mapsto (V \otimes V)[[x, x^{-1}]] \\ v &\mapsto \curlyvee(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v) x^{-k-1} \qquad (\text{where } \Delta_k(v) \in V \otimes V), \end{aligned}$$

$$c : V \mapsto \mathbb{C},$$

$$\rho : V \mapsto \mathbb{C},$$

called the *coproduct*, the *covacuum map* and the *co-Virasoro map*, respectively, satisfying the following 7 axioms:

1. Left counit: For all $v \in V$,

$$(c \otimes Id_V)\Lambda(x)v = v. \tag{2.12}$$

2. Cocreation: For all $v \in V$,

$$(Id_V \otimes c)\Lambda(x)v \in V[[x]] \quad \text{and} \tag{2.13}$$

$$\lim_{x \to 0}(Id_V \otimes c)\Lambda(x)v = v. \tag{2.14}$$

3. Truncation: Given $v \in V$, then $\Delta_k(v) = 0$ for k sufficiently small, or equivalently

$$\Lambda(x)v \in (V \otimes V)((x^{-1})). \tag{2.15}$$

4. Jacobi identity:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)(Id_V \otimes \Lambda(x_2))\Lambda(x_1) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)(T \otimes Id_V)$$
$$(Id_V \otimes \Lambda(x_1))\Lambda(x_2) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)(\Lambda(x_0) \otimes Id_V)\Lambda(x_2). \tag{2.16}$$

5. Virasoro algebra: The Virasoro algebra bracket,

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}d, \tag{2.17}$$

holds for all $j, k \in \mathbb{Z}$, where

$$(\rho \otimes Id_V)\Lambda(x) = \sum_{k \in \mathbb{Z}} L(k)x^{k-2}. \tag{2.18}$$

6. Grading: For each $k \in \mathbb{Z}$ and $v \in V_{(k)}$,

$$L(0)v = kv. \tag{2.19}$$

7. $L(1)$-derivative:

$$\frac{d}{dx}\Lambda(x) = (L(1) \otimes Id_V)\Lambda(x). \tag{2.20}$$

We denote this vertex operator coalgebra by (V, Λ, c, ρ) or simply by V when the structure is clear. A VOC *homomorphism* (or *morphism* in the category of VOCs) is a linear function f from one VOC $(U, \Lambda_U, c_U, \rho_U)$ to another VOC $(V, \Lambda_V, c_V, \rho_V)$ such that $c_V \circ f = c_U$ and

$$(f \otimes f)\Lambda_U(x) = \Lambda_V(x)f.$$

The truncation condition (2.15) for a vertex operator coalgebra warrants consideration analogous to that for the truncation condition for vertex operator algebras.

PROPOSITION 2.6. The weight condition

$$\text{wt } \Delta_k(v) = r + k + 1, \tag{2.21}$$

for $v \in V_{(r)}$, is a consequence of the Jacobi identity (2.16), grading (2.19), and the $L(1)$-derivative property (2.20).

As a result of the fact that Δ_k maps from V to $V \otimes V$, it bears mentioning that given $u \in V_{(r)}$ and $v \in V_{(s)}$ homogeneous vectors,

$$\text{wt } (u \otimes v) = r + s, \tag{2.22}$$

so the proposition states that Δ_k is a $k+1$ raising operator, mapping $V_{(n)}$ to $(V \otimes V)_{(n+k+1)}$ for each $n \in \mathbb{Z}$.

PROOF. We take the $x_0^{-1}x_1^{-2}$ coefficient of the Jacobi identity using (2.1).

$$\begin{aligned}
(Id_V \otimes \Lambda(x_2))\Delta_1 &- (T \otimes Id_V)(Id_V \otimes \Delta_1)\Lambda(x_2) \\
&= Res_{x_0} Res_{x_1} x_1 x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right)(\Lambda(x_0) \otimes Id_V)\Lambda(x_2) \\
&= Res_{x_0} Res_{x_1} x_1 x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right)(\Lambda(x_0) \otimes Id_V)\Lambda(x_2) \\
&= Res_{x_0}(x_2 + x_0)(\Lambda(x_0) \otimes Id_V)\Lambda(x_2) \\
&= x_2(\Delta_0 \otimes Id_V)\Lambda(x_2) + (\Delta_1 \otimes Id_V)\Lambda(x_2).
\end{aligned}$$

Next we compose with the map $\rho \otimes Id_V \otimes Id_V$ and, applying (2.18), we have the $L(0)$-commutation formula for VOCs:

$$(Id_V \otimes L(0))\Lambda(x_2) + (L(0) \otimes Id_V)\Lambda(x_2) = \Lambda(x_2)L(0) - x_2(L(1) \otimes Id_V)\Lambda(x_2). \tag{2.23}$$

Considering a homogeneous vector $v \in V_{(r)}$, we have wt $u = r$. Hence, substituting (2.20) into (2.23) and comparing coefficients, we have

$$\begin{aligned}
\text{wt } (\Delta_k(v))\ \Delta_k(v) &= ((Id_V \otimes L(0)) + (L(0) \otimes Id_V))\Delta_k(v) \\
&= \text{wt } (v)\ \Delta_k(v) - (-k-1)\ \Delta_k(v) \\
&= (r + k + 1)\ \Delta_k(v)
\end{aligned}$$

for each $k \in \mathbb{Z}$. □

COROLLARY 2.7. In the presence of the other axioms, the weight condition (2.21) may replace the truncation condition (2.15) in the definition of vertex operator coalgebra.

PROOF. We have shown that (2.21) is a consequence of the standard definition of VOC and now show the converse. Let $v \in V_{(r)}$. By positive energy, i.e. $V_{(n)} = 0$ for $n << 0$, if $\Delta_k(v)$ is nonzero, there is a fixed $N <$ wt $\Delta_k(v) = r + k + 1$. Then $-k - 1 < r - N$, i.e. $\Lambda(x)v \in Vx^{r-N}[[x^{-1}]]$. □

REMARK 2.8. As in Remark 2.4, either (2.15) or both positive energy and (2.21) are needed in the definition of VOC to make the Jacobi identity (2.16) well defined.

2.4. A correspondence between VOAs and VOCs. The following theorem demonstrates a correspondence between VOAs and VOCs. We will show that a VOA structure on a vector space V is equivalent to a VOC structure on the graded dual V' of V. This is the primary building block in demonstrating an equivalence of categories between VOAs and VOCs.

THEOREM 2.9. *Let $V = \coprod_{k\in\mathbb{Z}} V_{(k)}$ be a graded vector space. Choose two distinguished vectors $\mathbf{1} \in V_{(0)}$ and $\omega \in V_{(2)}$ and a linear map*

$$Y(\cdot, x) : V \to End\, V[[x, x^{-1}]].$$

Additionally, define $c \in V''_{(0)}$ to be the double dual of $\mathbf{1}$, $\rho \in V''_{(2)}$ to be the double dual of ω, and a linear operator $\Lambda(x) : V' \to (V' \otimes V')[[x, x^{-1}]]$ defined by

$$\langle \Lambda(x)u', v \otimes w\rangle = \langle u', Y(v, x)w\rangle$$

for all $u' \in V'$, $v, w \in V$. The quadruple $(V, Y, \mathbf{1}, \omega)$ is a vertex operator algebra if and only if the quadruple (V', Λ, c, ρ) is a vertex operator coalgebra.

PROOF. Certainly V has the properties $\dim V_{(k)} < \infty$ for all $k \in \mathbb{Z}$ and $V_{(k)} = 0$ for k sufficiently small precisely when V' has the analogous properties. In regard to the axioms for VOAs and VOCs, in fact, each axiom of a VOA (replacing truncation with the weight condition, cf. Corollary 2.3) is equivalent to the axiom of corresponding number for a VOC (replacing truncation with the weight condition, cf. Corollary 2.7):

1. Left unit and counit: For any $u' \in V'$, $v \in V$,

$$\begin{aligned}\langle (c \otimes Id_{V'})\Lambda(x)u', v\rangle &= \langle \Lambda(x)u', \mathbf{1} \otimes v\rangle \\ &= \langle u', Y(\mathbf{1}, x)v\rangle.\end{aligned}$$

Thus if either $(c \otimes Id_{V'})\Lambda(x)$ or $Y(\mathbf{1}, x)$ is the identity map, so is the other.

2. Creation and cocreation: For all $u' \in V'$, $v \in V$,

$$\begin{aligned}\langle (Id_{V'} \otimes c)\Lambda(x)u', v\rangle &= \langle \Lambda(x)u', v \otimes \mathbf{1}\rangle \\ &= \langle u', Y(v, x)\mathbf{1}\rangle.\end{aligned}$$

Again, if $(Id_{V'} \otimes c)\Lambda(x)u' \in V'[[x]]$ then $Y(v, x)\mathbf{1} \in V[[x]]$ and vice versa. Similarly,

$$\lim_{x\to 0} \langle (Id_{V'} \otimes c)\Lambda(x)u', v\rangle = \lim_{x\to 0} \langle u', Y(v, x)\mathbf{1}\rangle.$$

Hence, if either side equals $\langle u', v\rangle$, by the definition of Λ both sides equal $\langle u', v\rangle$.

3. Weight condition: For $u' \in V'_{(r)}$, $v \in V_{(s)}$, $w \in V_{(t)}$, and any $k \in \mathbb{Z}$, we have

$$\langle \Delta_k(u'), v \otimes w\rangle = \langle u', v_k w\rangle.$$

The weight condition on VOAs (2.11) implies that the right-hand side is zero unless $r = \text{wt } u' = \text{wt } v_k w = s + t - k - 1$. The weight condition on VOCs (2.21) implies that the left-hand side is zero unless $r + k + 1 = \text{wt } \Delta_k(u') = \text{wt } (v \otimes w) = s + t$. Thus they are equivalent.

4. Jacobi identity: For all $u' \in V'$, $v_1, v_2, v_3 \in V$

$$\langle (Id_V \otimes \Lambda(x_2))\Lambda(x_1)u', v_1 \otimes v_2 \otimes v_3\rangle = \langle \Lambda(x_1)u', v_1 \otimes Y(v_2, x_2)v_3\rangle = \langle u', Y(v_1, x_1)Y(v_2, x_2)v_3\rangle, \tag{2.24}$$

$$\begin{aligned}\langle (T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2)u', &v_1 \otimes v_2 \otimes v_3\rangle \\ &= \langle (Id_V \otimes \Lambda(x_1))\Lambda(x_2)u', v_2 \otimes v_1 \otimes v_3\rangle \\ &= \langle \Lambda(x_2)u', v_2 \otimes Y(v_1, x_1)v_3\rangle \\ &= \langle u', Y(v_2, x_2)Y(v_1, x_1)v_3\rangle,\end{aligned} \tag{2.25}$$

$$\langle (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)u', v_1 \otimes v_2 \otimes v_3\rangle = \langle \Lambda(x_2)u', Y(v_1, x_0)v_2 \otimes v_3\rangle = \langle u', Y(Y(v_1, x_0)v_2, x_2)v_3\rangle. \tag{2.26}$$

Equations (2.24), (2.25) and (2.26) make it clear that the VOA Jacobi identity (2.6) is equivalent to the VOC Jacobi identity (2.16). Note that showing (2.6) or (2.16) is well defined, given the same equation in brackets, requires the corresponding truncation condition (or weight and positive energy conditions).

5. Virasoro algebra: Let $u' \in V'$ and $v \in V$. Any operator $L(k)$ on V which has weight k (i.e. $L(k) : V_{(n)} \to V_{(n-k)}$ for all $n \in \mathbb{Z}$) yields an operator $L'(-k)$ on V' of weight $-k$ via

$$\langle L'(-k)u', v\rangle = \langle u', L(k)v\rangle.$$

The converse is also true. Hence, since

$$\langle (\rho \otimes Id_V)\Lambda(x)u', v\rangle = \langle \Lambda(x)u', \omega \otimes v\rangle = \langle u', Y(\omega, x)v\rangle, \tag{2.27}$$

having $Y(\omega, x) = \sum_{k\in\mathbb{Z}} L(k)x^{-k-2}$ implies that $(\rho \otimes Id_V)\Lambda(x) = \sum_{j\in\mathbb{Z}} L'(j)x^{j-2}$, and vice versa. The Virasoro bracket relations,

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}d,$$

and

$$[L'(j), L'(k)] = (j - k)L'(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}d,$$

also imply each other via (2.27).

6. Grading: Taking the coefficients of x^{-2} in Equation (2.27) shows that

$$\langle L'(0)u', v\rangle = \langle u', L(0)v\rangle$$

so gradation on V and V' correspond.

7. $L(-1)$ and $L'(1)$-Derivatives: Let $u' \in V'$, $v, w \in V$. We note the correspondence between $L(-1)$ and $L'(1)$ given by the x^{-1} coefficients of Equation (2.27), and conclude

$$\begin{aligned}\langle (L'(1) \otimes Id_V)\Lambda(x)u', v \otimes w\rangle &= \langle \Lambda(x)u', L(-1)v \otimes w\rangle \\ &= \langle u', Y(L(-1)v, x)w\rangle.\end{aligned}$$

But by definition

$$\frac{d}{dx}\langle u', Y(v,x)w\rangle = \frac{d}{dx}\langle \Lambda(x)u', v\otimes w\rangle,$$

which means that the $L(-1)$-derivative property and the $L'(1)$-derivative imply one another. □

The correspondence in Theorem 2.9 provides an injective mapping between objects in the category of VOAs and objects in the category of VOCs. (We could consider this correspondence a bijection by identifying V and V'' with identical structures, but this would slightly change the objects in our categories.) Given a morphism f from object U to object V in the category of VOAs, certainly we have a linear function f' from the VOC W' to the VOC V' given by

$$\langle f'(v'), u\rangle = \langle v', f(u)\rangle \tag{2.28}$$

for every $v' \in V'$, $u \in U$. Further, given $v' \in V'$ and using the definitions of c, f' and f,

$$\begin{aligned} c\ f'(v') &= \langle f'(v'), \mathbf{1}\rangle \\ &= \langle v', f(\mathbf{1})\rangle \\ &= \langle v', \mathbf{1}\rangle \\ &= c(v'). \end{aligned}$$

Finally, for all $v' \in V'$, $u_1, u_2 \in U$ we observe that using the definitions of Λ, f' and f,

$$\begin{aligned} \langle (f'\otimes f')\Lambda(x)v', u_1\otimes u_2\rangle &= \langle \Lambda(x)v', f(u_1)\otimes f(u_2)\rangle \\ &= \langle v', Y(f(u_1),x)f(u_2)\rangle \\ &= \langle v', f(Y(u_1,x)u_2)\rangle \\ &= \langle f'(v'), Y(u_1,x)u_2\rangle \\ &= \langle \Lambda(x)f'(v'), u_1\otimes u_2\rangle. \end{aligned}$$

Thus, f' is a morphism in the category of VOCs and we have a contravarient functor from VOAs to VOCs. Similarly, a morphism in the category of VOCs gives rise to a morphism in the category of VOAs. We observe the following corollary.

Corollary 2.10. The category of vertex operator algebras and the category of vertex operator coalgebras are equivalent.

The functions between object sets described in Theorem 2.9 along with the functions between morphism sets defined in (2.28) establish this equivalence of categories. Were the categories of VOAs and VOCs defined in such a way that V was identified with the graded dual of its graded dual V'' this would be an isomorphism of categories, but since V and V'' are only canonically isomorphic we have an equivalence of categories.

Remark 2.11. The results in [**Hub3**] show that given a VOA V with a nondegenerate, Virasoro preserving bilinear form, V may be naturally endowed with a VOC structure. Notice that a nondegenerate, Virasoro preserving bilinear form amounts to a Virasoro invariant isomorphism between V and V'. Interpreting this result in light of Theorem 2.9, we see that V' has a natural VOC structure and the

Virasoro invariant isomorphism simply transports this structure to V. Such a bilinear form allows V to be viewed as a VOA and a VOC. This gives rise to the question of what a vertex bialgebra structure should entail. In [**L4**], Li integrates a single comultiplication operator into the structure of a VOA, but a bialgebra structure which fully subsumes the VOA and VOC structures has yet to be defined.

3. Modules and Comodules

3.1. Differing notions of modules.

DEFINITION 3.1. A *weak module* M for a VOA V is a vector space M equipped with a linear map $V \otimes M \to M[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} Y : V &\mapsto (\text{End } M)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{k \in \mathbb{Z}} v_k x^{-k-1} \qquad (\text{where } v_n \in \text{End } M), \end{aligned}$$

satisfying the following axioms:

1. Left unit: For all $w \in M$,

$$Y(\mathbf{1}, x)w = w. \tag{3.1}$$

2. Truncation: For all $v \in V$, $w \in M$, we require $v_k w = 0$ for k sufficiently large, or equivalently,

$$Y(v, x)w \in V((x)). \tag{3.2}$$

3. Jacobi Identity: For all $u, v \in V$, $w \in M$,

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1)w \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2)w. \end{aligned} \tag{3.3}$$

(where the operator $Y(u, x_0)$ is an operation on V itself, but every other Y is an operation on the module.)(cf. [**DLM**]).

REMARK 3.2. Though originally included in the definition, it was proven in [**DLM**] that the Virasoro relations, and the $L(-1)$-derivative property are consequences of the other axioms and may be omitted. Specifically, we have the property that given V a VOA of rank d and M a weak module of V, the module operation $Y(-, x)$ satisfies

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k} d, \tag{3.4}$$

for $j, k \in \mathbb{Z}$, where

$$Y(\omega, x) = \sum_{k \in \mathbb{Z}} L(k)x^{-k-2}, \tag{3.5}$$

and

$$\frac{d}{dx}Y(v,x) = Y(L(-1)v,x) \tag{3.6}$$

for all $v \in V$.

DEFINITION 3.3. A *generalized module* for a VOA V is a weak module endowed with a $\mathbb{C}$-graded vector space structure

$$M = \coprod_{\lambda \in \mathbb{C}} M_{(\lambda)}$$

such that

$$L(0)w = \lambda w, \text{ for } w \in V_{(\lambda)}$$

(cf. [**HL2**]). We say that the weight of such a w is λ.

DEFINITION 3.4. An *(ordinary) module* for a VOA V is a generalized module $M = \coprod_{\lambda \in \mathbb{C}} M_{(\lambda)}$ such that $\dim M_\lambda < \infty$ for $\lambda \in \mathbb{C}$ and $M_{\lambda+n} = 0$ for any fixed λ and $n \in \mathbb{Z}$ sufficiently small (cf. [**FHL**]).

Following Proposition 2.2 and Corollary 2.3, we consider a weight condition as a replacement for the truncation condition in the axioms of a vertex operator algebra module. In fact, the proofs of Proposition 2.2 and Corollary 2.3 may be interpreted in the module context as follows.

PROPOSITION 3.5. The weight condition

$$\text{wt } (v_k w) = r + s - k - 1 \qquad \text{for } v \in V_{(r)},\ w \in M_{(s)} \tag{3.7}$$

may replace the truncation condition in the definition of an (ordinary) module for a vertex operator algebra.

REMARK 3.6. Equation (3.7) is true for generalized modules but may not replace truncation as an axiom, for without positive energy Corollary 2.3 does not hold in the module context and the Jacobi identity is not well defined. Since Equation (3.7) requires a graded vector space (the notion of weight is inherently graded), it does not generalize to weak modules.

3.2. Differing notions of comodules.

DEFINITION 3.7. A *weak comodule* M for a given VOC V is a vector space equipped with a linear map

$$\begin{aligned} \mathit{\Lambda} : M &\mapsto (V \otimes M)[[x, x^{-1}]] \\ u &\mapsto \mathit{\Lambda}(x)u = \sum_{k \in \mathbb{Z}} \Delta_k(u) x^{-k-1} \end{aligned}$$

satisfying the following axioms:

1. Left counit: For all $w \in M$,

$$(c \otimes Id_V)\mathit{\Lambda}(x)w = w. \tag{3.8}$$

2. Truncation: Given $w \in M$, then $\Delta_k(w) = 0$ for k sufficiently small, or equivalently

$$\mathit{\Lambda}(x)w \in (V \otimes V)((x^{-1})). \tag{3.9}$$

3. Jacobi identity:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)(Id_V\otimes\varLambda(x_2))\varLambda(x_1)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)(T\otimes Id_V)$$
$$(Id_V\otimes\varLambda(x_1))\varLambda(x_2)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(\varLambda(x_0)\otimes Id_V)\varLambda(x_2) \tag{3.10}$$

where the operator $\varLambda(x_0)$ is an operation on V itself, but every other $\varLambda$ is an operation on the module.

4. Virasoro algebra: The Virasoro algebra bracket,

$$[L(j),L(k)]=(j-k)L(j+k)+\frac{1}{12}(j^3-j)\delta_{j,-k}d, \tag{3.11}$$

holds for $j,k\in\mathbb{Z}$, where for all $w\in M$

$$(\rho\otimes Id_V)\varLambda(x)w=\sum_{k\in\mathbb{Z}}L(k)x^{k-2}w. \tag{3.12}$$

5. $L(1)$-derivative:

$$\frac{d}{dx}\varLambda(x)=(L(1)\otimes Id_V)\varLambda(x). \tag{3.13}$$

Definition 3.8. A *generalized comodule* for a VOC V is a weak module endowed with a $\mathbb{C}$-graded vector space

$$M=\coprod_{\lambda\in\mathbb{C}}M(\lambda)$$

such that

$$L(0)w=\lambda w=(\text{wt } w)w, \text{ for } w\in V_{(\lambda)}.$$

Definition 3.9. An *(ordinary) comodule* for a VOC V is a generalized module $M=\coprod_{\lambda\in\mathbb{C}}M_{(\lambda)}$ such that $\dim M_\lambda<\infty$ for $\lambda\in\mathbb{C}$ and $M_{\lambda+n}=0$ for a fixed λ and $n\in\mathbb{Z}$ sufficiently small (cf. [**Hub4**]).

Again, the proof Proposition 2.6 and Corollary 2.7 generalize to the following:

Proposition 3.10. The weight condition

$$\text{wt } \Delta_k(w)=r+k+1 \qquad \text{for } w\in M_{(r)} \tag{3.14}$$

may replace the truncation condition in the definition of an (ordinary) comodule for a vertex operator coalgebra.

As in the case of modules, Equation (3.14) is true for generalized comodules but may not replace truncation as an axiom. It does not apply to weak modules (cf. Remark 3.6).

3.3. A correspondence between VOA modules and VOC comodules. In view of the fact that a correspondence between a VOA structure on V and a VOC structure on the graded dual V' (or a VOC structure on V and a VOA structure on V') has been established, it is natural to investigate this correspondence as it pertains to modules and comodules. Given a weak module, a generalized module or an ordinary module M over a VOA V, what may be said of M's structure over V' as a VOC?

For weak modules the situation is bleak indeed. The correspondence between VOAs and VOCs is given by considering the graded dual. However, given M a weak module over a VOA V, we have no natural gradation on M. Thus the only notion of dual is the standard one: $M^* = \mathrm{Hom}(M, \mathbb{C})$. (If M were in fact graded, this would amount to considering the closure of the graded dual, $\overline{M'}$.) It is reasonable to ask whether M^* has an induced weak comodule structure over the VOC V' and instructive to see how this fails to be true in a counterexample. Notice that cocreation, the Virasoro algebra and the L(1)-derivative property do hold on M^* following just as in the proof of Theorem 2.9. Were truncation to hold, the Jacobi identity would follow as in Theorem 2.9 as well. Truncation, however, fails to transfer in general from weak modules to their duals.

COUNTEREXAMPLE 3.11. Let $V = \mathbb{C}[z_1, z_2, z_3, \ldots]$, which is a presentation of the one dimensional Heisenberg algebra (see [**D**]) with $\mathbf{1} = 1$, $\omega = \frac{1}{2}z_1^2$ and

$$Y(z_1, x) = \sum_{k\in\mathbb{Z}_+}\left(z_k x^{k-1} + k\frac{\partial}{\partial z_k}x^{-k-1}\right). \tag{3.15}$$

V may be viewed as a weak module over itself. (V viewed an (ordinary) module over itself is generally called the adjoint module.) We will show that V^* is not even a weak comodule over the vertex operator coalgebra V'. There exists an element $u^* \in V^*$ such that $u^*(z_i) = 1$ for all $i \in \mathbb{Z}_+$. But then $\Lambda(x)u^*$ does not satisfy the truncation condition. Visually, for all $n \in \mathbb{Z}_+$,

$$\mathrm{Res}_x x^{-n}\langle \Lambda(x)u^*, z_1 \otimes \mathbf{1}\rangle = \mathrm{Res}_x x^{-n}\langle u^*, Y(z_1, x)\mathbf{1}\rangle \tag{3.16}$$

$$= \mathrm{Res}_x x^{-n}\langle u^*, \sum_{k\in\mathbb{Z}_+}\left(z_k x^{k-1} + k\frac{\partial}{\partial z_k}x^{-k-1}\right)\mathbf{1}\rangle \tag{3.17}$$

$$= \langle u^*, z_n\rangle \ = 1. \tag{3.18}$$

Thus V^* fails to be a V'-comodule.

In the case of generalized modules there is perhaps stronger argument for expecting to find an induced V'-comodule given a VOA V and a generalized module M. After all, we now have a natural notion of graded dual $M' = \coprod_{k\in\mathbb{Z}}\left(W_{(k)}\right)'$ on which to hope such a structure would coalesce. Again, however, we fall short based on the truncation condition.

COUNTEREXAMPLE 3.12. Let $\mathcal{L} = \coprod_{j\in\mathbb{Z}}\mathbb{C}L_j \coprod \mathbb{C}c$ be the Virasoro algebra and define the following subalgebras as follows:

$$\mathcal{L}_{(-)} = \coprod_{n\in\mathbb{Z}_+}\mathbb{C}L_n,$$

$$\mathcal{L}_{(0)} = \mathbb{C}L_0 \oplus \mathbb{C}c,$$

$$\mathcal{L}_{(\leq 1)} = \mathcal{L}_{(-)} \oplus \mathcal{L}_{(0)} \oplus \mathbb{C}L_{-1}.$$

Considering a complex number, ℓ, we construct an $\mathcal{L}_{(\leq 1)}$-module structure on $\mathbb{C}$ by defining $\mathcal{L}_{(-)} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_{-1}$ to act trivially on $\mathbb{C}$ and c to act as the scalar ℓ. Denote this module by $\mathbb{C}_\ell$. We now form the induced Virasoro module

$$V = V_{Vir}(\ell, 0) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(\leq 1)})} \mathbb{C}_\ell$$

where U is the universal enveloping algebra. By the Poincaré-Birkhoff-Witt theorem, $V_{Vir}(\ell, 0) = U(\coprod_{n\geq 2} \mathbb{C}L_{-n})$ as vector spaces and has a basis consisting of vectors of the form

$$L(-m_1)L(-m_2)\cdots L(-m_r)\mathbf{1}_\ell$$

where $r \geq 0$, $m_1 \geq m_2 \geq \cdots \geq m_r \geq 2$ (and $\mathbf{1}_\ell = 1 \otimes 1_\ell$). This element has weight $m_1 + m_2 + \cdots + m_r$. V is a vertex operator algebra with vacuum vector $\mathbf{1}_\ell$ and vacuum vector $L_{-2}\mathbf{1}_\ell$ (cf. Theorem 6.1.5 in [**LL**]).

Next, considering two complex number, ℓ and h, we construct a $(\mathcal{L}_{(-)} \oplus \mathcal{L}_{(0)})$-module structure on $\mathbb{C}$ by defining $\mathcal{L}_{(-)}$ to act trivially on $\mathbb{C}$, c to act as the scalar ℓ and L_0 to act as the scalar h. Denote this module by $\mathbb{C}_{\ell,h}$. We form the induced Virasoro module

$$M_{Vir}(\ell, h) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(-)}\oplus\mathcal{L}_{(0)})} \mathbb{C}_{\ell,h}.$$

By Theorem 6.1.9 in [**LL**], $M_{Vir}(\ell, h)$ is an (ordinary) V-module where $M_{Vir}(\ell, h)_{(n+h)}$, the L_0-eigenspace with eigenvalue $n + h$, has a basis consisting of elements of the form

$$L(-m_1)L(-m_2)\cdots L(-m_r)\mathbf{1}_{\ell,h}$$

where $r \geq 0$, $m_1 \geq m_2 \geq \cdots \geq m_r \geq 2$, and $m_1 + m_2 + \cdots + m_r = n$.

Let $M = \coprod_{k\in\mathbb{N}} M_{Vir}(0, -k)$. M is still a restricted module for the Virasoro algebra and hence still a generalized vertex operator algebra module (cf. Theorem 6.1.4 in [**LL**]). However, M fails both of the grading conditions to be an ordinary module. We will show that M' also fails the truncation condition for VOC comodules.

Given any $j \in \mathbb{Z}_-$, we know that for all $k \in \mathbb{Z}_+$ we have $(L(-k)\mathbf{1}_{\ell,j-k}) \in M_{(j)}$. Let $\gamma \in (M_{(j)})'$ such that $\gamma(L(-k)\mathbf{1}_{\ell,j-k}) = 1$ for all $k \in \mathbb{Z}_+$. Then for all $k \in \mathbb{Z}_+$ we have

$$\begin{aligned}
\mathrm{Res}_x x^{-k+1}\langle \Lambda(x)\gamma, \omega \otimes \mathbf{1}_{\ell,j-k}\rangle &= \mathrm{Res}_x x^{-k+1}\langle \gamma, Y(\omega, x)\mathbf{1}_{\ell,j-k}\rangle \\
&= \mathrm{Res}_x x^{-k+1}\langle \gamma, \sum_{n\in\mathbb{Z}} L(n)\mathbf{1}_{\ell,j-k}x^{-n-2}\rangle \\
&= \langle \gamma, L(-k)\mathbf{1}_{\ell,j-k}\rangle \\
&= 1
\end{aligned}$$

Therefore, M' fails the truncation condition.

Finally, we are left with the question, given an ordinary module M over a VOA V, is there an induced (ordinary) comodule structure? The answer is yes.

Theorem 3.13. *Let M be a module over a VOA V. Then the graded dual M' is a V' VOC comodule where the coproduct $\Lambda(x) : M' \to (V' \otimes M')[[x, x^{-1}]]$ is defined by*

$$\langle \Lambda(x)m', v \otimes m\rangle = \langle m', Y(v, x)m\rangle.$$

for all $m' \in M'$, $v \in V$, $m \in M$.

Similarly, if M is a comodule over a VOC V, then M' is a module over the VOA V' where the product $Y(\cdot,x) : V' \to (End\ M')[[x,x^{-1}]]$ is defined by

$$\langle Y(v',x)m',m\rangle = \langle v' \otimes m', \varLambda(x)m\rangle$$

for all $v' \in V'$, $m' \in M'$, $m \in M$.

This Proposition is proved similarly to Theorem 2.9. On axiom 3, Propositions 3.5 and 3.10 take the place of Corollaries 2.3 and 2.7, and the rest of the axioms follow nearly verbatim. The converse (comodule to module) is similar.

Remark 3.14. It is interesting to note that any generalized module which satisfies the positive energy condition, $M_{\lambda+n} = 0$ for a fixed $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}$ sufficiently small, adheres to the comodule truncation condition just as in Corollary 2.7 and, hence, its dual is a comodule. It may also be the case that any module satisfying finite dimensionality of eigenspaces, dim $M_\lambda < \infty$ for all $\lambda \in \mathbb{C}$, also satisfies truncation. (The author has been unable to establish this.) Such a result seems plausible since failing comodule truncation would imply infinitely many vectors from infinitely many negative weight spaces had all been raised to the same nonzero vector. It is interesting to observe how the properties of positive energy and finite dimensionality of subspaces both contribute to the failure of Counterexample 3.12.

3.4. Contragredient modules. Classically, given a VOA V and an (ordinary) module M, the graded dual M' may be also be viewed as a V-module (cf. [**FHL**], [**L1**]) under the vertex operator map $Y' : V \to (\text{End } M')[[x,x^{-1}]]$ given by

$$\langle Y'(v,x)m',m\rangle = \langle m', Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})m\rangle \tag{3.19}$$

for $v' \in V'$, $m' \in M'$, $m \in M$. Applying Theorem 3.13 to the V-module M', we obtain a V'-comodule structure on $M'' = M$. Thus if one starts with a VOA V and a module M, one can view M as both a module and a comodule and also view M' as a module and a comodule.

Similarly, if one begins with a VOC V and a comodule M, Theorem 3.13 implies that M' is a module over the VOA V'. Again we may take the contragredient to obtain $M'' = M$ as a VOA module over V', and we may employ Theorem 3.13 yet again to obtain M' as a VOC comodule over V. Thus, as before one can view M as both a module and a comodule and also view M' as a module and a comodule.

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P.O. Box 13040, Department of Mathematics and Statistics, Stephen F. Austin State University, Nacogdoches, TX 75962

E-mail address: `hubbardke@sfasu.edu`