LECTURES FOR WEEK 3

CUBIC-FOOT VOLUME OF A LOG

Ways to calculate cubic foot volume

1) **xylometer**: a tub of water – submerge tree or log in water and find volume of water displaced.

2) **graphic**:

   example: log length = 14 feet, each section 2 feet in length, total number of cross-sections = 7

<table>
<thead>
<tr>
<th>Section</th>
<th>Dib (inches)</th>
<th>Cross-Setional area (sq. ft.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>28.7</td>
<td>4.5</td>
</tr>
<tr>
<td>2</td>
<td>28.7</td>
<td>4.5</td>
</tr>
<tr>
<td>3</td>
<td>28.7</td>
<td>4.5</td>
</tr>
<tr>
<td>4</td>
<td>27.1</td>
<td>4.0</td>
</tr>
<tr>
<td>5</td>
<td>27.1</td>
<td>4.0</td>
</tr>
<tr>
<td>6</td>
<td>25.3</td>
<td>3.5</td>
</tr>
<tr>
<td>7</td>
<td>25.3</td>
<td>3.5</td>
</tr>
<tr>
<td>8</td>
<td>23.5</td>
<td>3.0</td>
</tr>
</tbody>
</table>

For our example, 1 sq. inch on graph = 2 cubic feet.

Use a dot grid or planimeter to get the square inches of the graphed log.

Then, the cubic foot of the log = sq. inches from dot grid * 2 cubic feet
3) analytic:

Cone: \[ \text{Volume} = \frac{1}{3} \times Ab \times H \]

Cylinder: \[ \text{Volume} = Ab \times H \]

Neiloid: \[ \text{Volume} = \frac{1}{4} \times Ab \times H \]

Paraboloid: \[ \text{Volume} = \frac{1}{2} \times Ab \times H \]

How are these volume equations derived?

Rotate the line/curve around the X axis

\[ Y = k \]

: Cylinder

\[ Y = k \times X^{1/2} \]

: Paraboloid
\[ Y = k \times X \quad : \text{Cone} \]

\[ Y = k \times X^{3/2} \quad : \text{Neiloid} \]
Derive a Parabolic Volume Equation

Here, the area of each slice is, \[ \text{Area} = \pi \cdot r^2 \] and, \[ r = k \cdot X^{1/2} \]

Therefore, \[ 
\text{Area} = \pi \cdot (k \cdot X^{1/2})^2 = \pi k^2 X 
\]

**Frustrum of cones, cylinders, etc.**

A frustrum is a piece of something, like a cone, cylinder, etc.

In the derivation, the integral becomes:

\[
V = \int_0^b \pi k^2 X dX = \pi k^2 \int_0^b X dX
\]

\[
= \pi k^2 \cdot \frac{X^2}{2} \bigg|_0^b = \frac{\pi k^2}{2} \cdot b^2 - 0
\]

\[
= \pi k^2 \cdot \frac{b}{2}
\]

\[
= Ab \cdot \frac{\text{length}}{2}
\]
Frustrum Formulae

Paraboloids:  
1.) Smalian’s: \( V = \frac{1}{2}(Ab + Au) \cdot L \)

2.) Huber’s: \( V = Am \cdot L \)

Smalian’s and Huber’s are exactly equal if the log is a true paraboloid

Cones: \( V = \frac{1}{3}(Ab + \sqrt{Ab \cdot Au} + Au) \cdot L \)

Neiloids: \( V = \frac{1}{4}(Ab + \sqrt[3]{Ab^2 \cdot Au} + \sqrt[3]{Ab \cdot Au^2} + Au) \cdot L \)

Newton’s: This is the most accurate formula, and it holds for neiloids, cones, and paraboloids

\( V = \frac{1}{6}(Ab + 4Am + Au) \cdot L \)

Comparison of Formulae

32 feet

Smalian Overestimates  Huber Underestimates

Dib=16“  Dib=12“  Dib=8“

Neiloid Underestimates

These are straight lines; a conic section

Remember to use inside bark diameters (dib) to calculate the areas to use in the volume formulas.
Taper is the change in diameter with length.

For our example, \( \text{Taper} = \frac{16'' - 8''}{32'} = 4'' \text{ taper in 16'} \)

\[
\begin{align*}
\text{Ab} &= 0.005454*D^2 = 1.40 \text{ sq. ft.} \\
\text{Au} &= 0.35 \text{ sq. ft.}
\end{align*}
\]

So,

\[
\begin{align*}
\text{Cone:} & \quad V = \frac{1}{3}(\text{Ab} + \sqrt{\text{Ab} \times \text{Au} + \text{Au}}) \times L = \frac{1}{3}(1.4 + \sqrt{1.4 \times 0.35 + 0.35}) \times 32' = 26 \text{ ft}^3 \\
\text{Smalian’s:} & \quad V = \frac{1}{2}(\text{Ab} + \text{Au}) \times L = \frac{1}{2}(1.4 + 0.35) \times 32' = 28 \text{ ft}^3 \\
\text{Huber’s:} & \quad V = \text{Am} \times L = 0.79 \text{ ft}^2 \times 32' = 25 \text{ ft}^3 \\
\text{Newton’s:} & \quad V = \frac{1}{6}(\text{Ab} + 4\text{Am} + \text{Au}) \times L = \frac{1}{6}(1.4 + 4 \times 0.79 + 0.35) \times 32' = 26 \text{ ft}^3
\end{align*}
\]

**Cord Volume**

A *cord* is a unit of volume that equals 128 cubic feet. This is equal to a stack of wood with the dimensions 4 x 4 x 8 feet, which includes the wood, bark, and empty space. This is a common measure of volume for pulpwood in the South. Avery and Burkhart on page 57 list some common specifications for pulpwood. Another measure of volume for pulpwood that you may encounter in the South is a *cunit*, which equals 100 cubic feet of solid wood. Cunits are not commonly used today.
**TREE FORM**

1.) **Taper Equations**

A taper equation mathematically describes the stem profile: \( \frac{d}{DBH} = f\left(\frac{h}{H}\right) \)

Thus, relative diameter and relative height are related. Taper equations can be used to “cut” a standing tree into logs to find their individual volumes.

**Example:**

Let’s assume that the cross-sectional area of a stem is linearly related to the height.

1.) For a given tree, only the area of the base, “a”, is known.

2.) Find the slope, “b”:

\[
b = \frac{(Y_2 - Y_1)}{(X_2 - X_1)} = \frac{(0 - Q)}{(H - 4.5)} = \frac{-Q}{(H - 4.5)}
\]
3.) Find the expression for “a”:

\[
b = \frac{(Y_2 - Y_1)}{(X_2 - X_1)} = \frac{(Q - a)}{(4.5 - 0)}
\]

\[
b \times 4.5 = Q - a
\]

\[
a + b \times 4.5 = Q
\]

\[
a = Q - b \times 4.5
\]

In General,

\[
Q = BA \text{ at DBH} \\
H = \text{total tree height} \\
A = \text{cross-sectional area at } h
\]

\[
h = \text{height on bole at d} \\
d = \text{diameter at } h
\]

\[
b = \frac{-Q}{(H - 4.5)}
\]

\[
a = Q - b \times 4.5
\]

Since we now know that \( A = a + b \times h = 0.005454 \times d^2 \), we can find expressions for \( d \) and \( h \):

\[
d = \sqrt{\frac{(a + b \times h)}{0.005454}}
\]

\[
h = \frac{(0.005454 \times d^2 - a)}{b}
\]

We can also find the volume equation by integration:

\[
V = \int A \, dh
\]

**Let’s look at a numerical example…**

Let \( Q = 1 \, \text{ft}^2 \) and \( H = 104.5 \, \text{feet} \). Then,

\[
b = \frac{-1}{(104.5 - 4.5)} = -0.01
\]

\[
a = 1 - (-0.01 \times 4.5)
\]
Then,

\[ A = 1.045 - 0.01h \]

Now, answer some questions:

1.) Estimate the diameter (d in inches) at h = 50 ft.

\[ A = 1.045 - 0.01(50') = 0.545 \text{ ft}^2 \]

\[ A = 0.005454*d^2 \]

\[ 0.545 = 0.005454*d^2 \]

\[ d^2 = 99.9267 \]

\[ d = 9.996 \text{ or } 10 \text{ inches} \]

2.) Estimate h where d = 4 inches

\[ A = 0.005454*d^2 = 0.005454*4^2 = 1.045 - 0.01*h \]

\[ 0.0873 = 1.045 - 0.01*h \]

\[ 0.01*h = 1.045 - 0.0873 \]

\[ h = 95.8 \text{ feet} \]

3.) Estimate volume to a d = 4 inch top:

\[ V = \int (1.045 - 0.01*h)dh = 54.2 \text{ ft}^3 \]

4.) Cut tree into 16 foot logs (assume a 6 inch trim allowance and a 1 foot stump).
2.) Form Factors & Quotients

A form factor is the ratio of actual tree volume to the volume of a specific geometric solid. Since upper-stem diameters are necessary to compute these volumes, stem form is usually described by the diameters themselves. This expression is called a form quotient, which is the ratio of an upper-stem diameter to DBH. A common form quotient used in many regions is Girard form class. Girard form class is the ratio between the diameter inside bark at the top of the first 16 foot log and DBH.

\[
\text{Girard form class} = \frac{\text{DIB}_{17.3}}{\text{DBH}}
\]